The holonomy representation, II.

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Recall where we had reached last time.

A principal bundle or principal G-bundle (G a Lie group) is a fibre bundle $(E, \pi, B; G)$ such that the fibres $E_b = \pi^{-1}(b)$ have a free and transitive G-action, and the local trivializations are G-equivariant:

So $\varphi : U \times G \longrightarrow \pi^{-1}(U)$ must satisfy

$\varphi(b, g)h = \varphi(b, gh)$

Example: If E is a smooth manifold with a G-action that is free and proper $\Longrightarrow$ E/G is a smooth manifold and $E \longrightarrow E/G \simeq B$ is the map.

Def: A connection on $(E, \pi, B; G)$ consists of a smooth assignment $p \longmapsto \mathcal{H}_p \subset T_p E$, such that

① $T_p E \simeq T_p E_b \oplus \mathcal{H}_p$ (here $\pi(p) = b$)

(So the chosen subspace is transversal to the tangent space of the fibre)

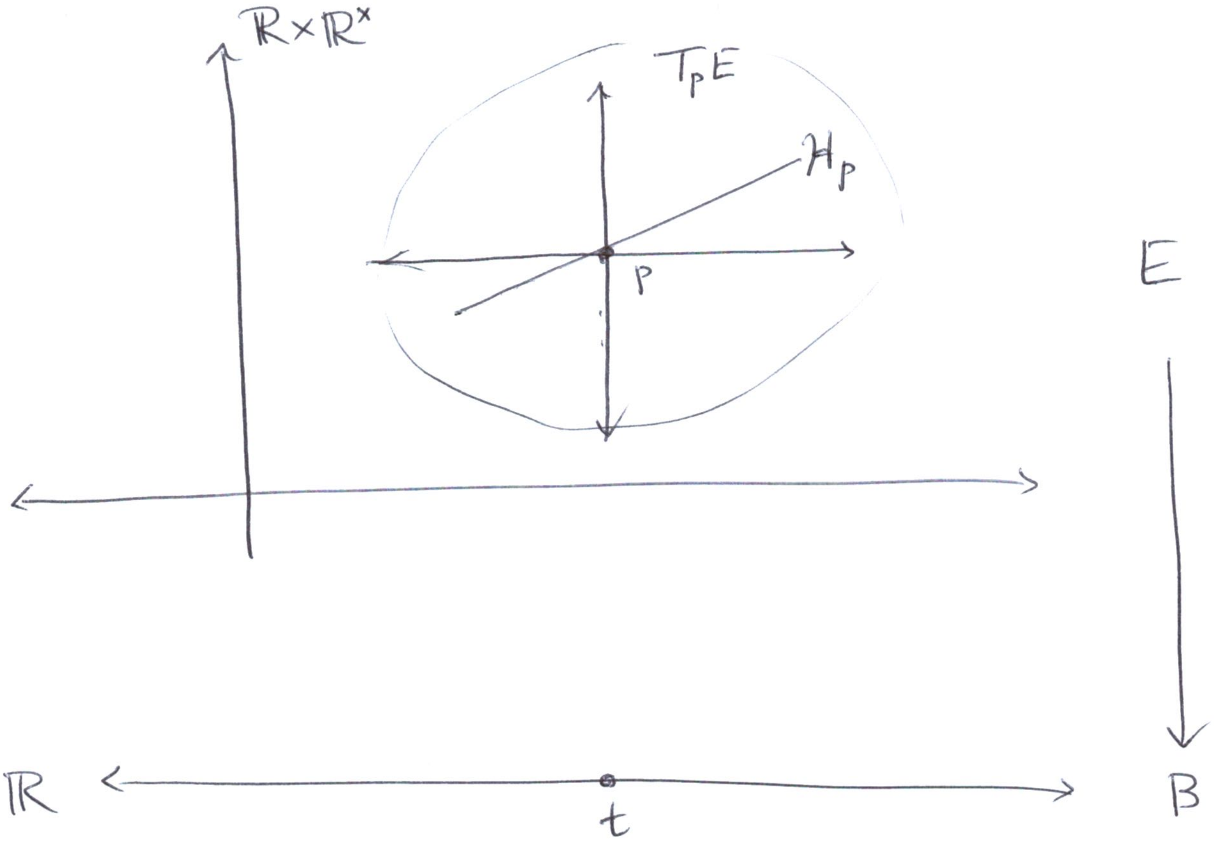
② G-invariance: $(R_g)_* (\mathcal{H}_g) = \cancel{R_g p}\ \mathcal{H}_{p \cdot g}$

Here, $R_g(p) = p \cdot g$ and $(R_g)_*$ is the differential $(dR_g)_p$.

Example: $E = \mathbb{R} \times \mathbb{R}^{\times}$ (here $\mathbb{R}^{\times}$ is nonzero reals, thinking of it as $GL(1,\mathbb{R})$). Then $B = \mathbb{R}$ and $\pi : \mathbb{R} \times \mathbb{R}^{\times} \longrightarrow \mathbb{R}$ is pr_1, so $\pi(x,y) = x$. Then the action of $\mathbb{R}^{\times}$ on the bundles are ~~translation~~ multiplication by an element of $\mathbb{R}^{\times}$, and $(t,a)\cdot b = (t,ab)$ is how we express this. So $R_b(t,a) = (t,ab)$.

What is the connection?

$T_{(t,a)}E = \mathbb{R} \times \mathbb{R}$ and at (t,a) the tangent space is $\left\langle \frac{\partial}{\partial t}\Big|_{(t,a)}, \frac{\partial}{\partial t}\Big|_{(t,a)} \right\rangle$. In pictures:



So a connection is specified by

$$\mathcal{H}_p = \mathrm{Span}\left\langle \frac{\partial}{\partial t}\Big| + g(t,a)\frac{\partial}{\partial a} \right\rangle,$$

this is from ① which requires transversality.

Then condition ②, G-invariance, gives

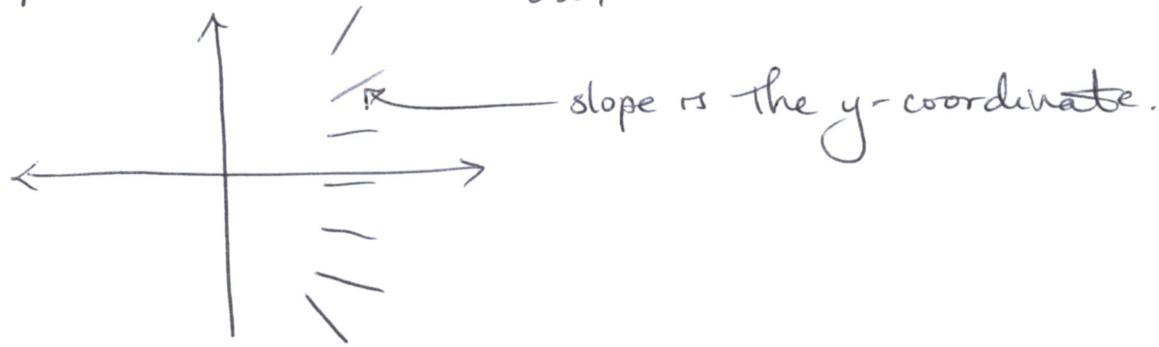
$$(R_b)_* \left(\frac{\partial}{\partial t}\bigg|_{(t,a)} + g(t,a) \frac{\partial}{\partial a}\bigg|_{(t,a)} \right) = \frac{\partial}{\partial t} + g(t,ab) \frac{\partial}{\partial a}$$

$$\parallel$$

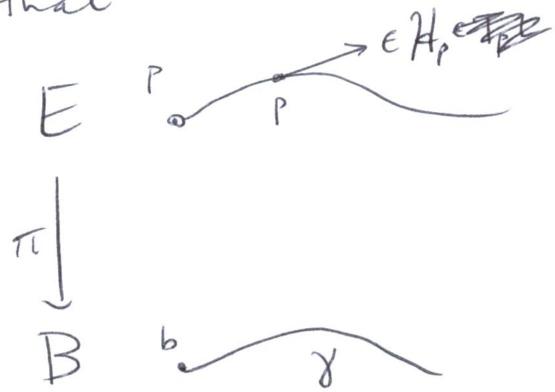
$$\frac{\partial}{\partial t}\bigg|_{(t,ab)} + g(t,a) \left(b \frac{\partial}{\partial a}\bigg|_{(t,ab)} \right)$$

So we need $g(t,a) \cdot b = g(t,ab)$ So let $f(t) = g(t,1)$ and see then that any connection is described by $\mathcal{H}_{(t,a)} = \text{span}\left\langle \frac{\partial}{\partial t} + f(t)\, a \frac{\partial}{\partial a} \right\rangle$ where f is any function.

In pictures, suppose e.g. $f(t) = 1$, and $\mathcal{H} = \text{span}\left\langle \frac{\partial}{\partial t} + a \frac{\partial}{\partial a} \right\rangle$ and we get



Now as mentioned last time, what are connections good for? Well, you can lift a path in B to a path in E, so that



So given $\gamma(t) \in B$ and $p \in E_{\gamma(0)}$, $\exists!$ $\tilde{\gamma}(t) \subseteq E$ such that

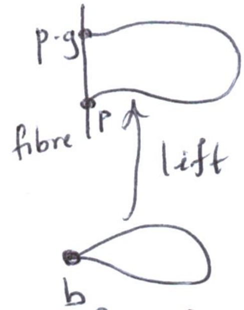
$$\frac{d}{dt} \tilde{\gamma}(t) \in H_{\tilde{\gamma}(t)}$$

We also have parallel transport $E_{\gamma(0)} \to E_{\gamma(1)}$ defined by "take endpoints of horizontal lifts".

<u>Definition</u>: A <u>flat</u> connection on (E, π, B, G) is a connection $\mathcal{H}$ such that parallel transport depends only on homotopy class of the path with fixed endpoints.

(So endpoints of lifts of homotopic paths depend only on the homotopy type of path. Note this is similar to covering spaces)

In particular, if $(E, \mathcal{H})$ is a flat G bundle we get $\pi_1(B, b) \longrightarrow G$ given by



, there's (fix p).

a unique g with $p \cdot g$ as endpoint of the lift. This map is denoted

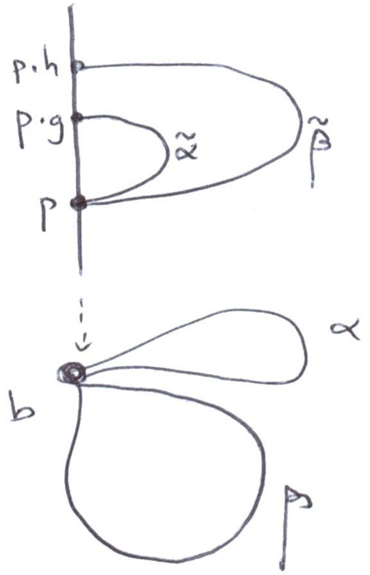
$$\mathrm{Hol}_p^{\mathcal{H}} : \pi_1(B,b) \longrightarrow G$$

given by ~~Hol~~ $\mathrm{Hol}_p^{\mathcal{H}}(\gamma) = g^{-1}$ ← inverse here due to left/right actions.

where $\tilde{\gamma}(1) = p \cdot g$.

<u>Claim</u>: $\mathrm{Hol}_p^{\mathcal{H}}$ is a group homomorphism.

<u>Proof</u>: Given $\alpha, \beta \in \pi_1(B,b)$ consider the picture:



where $\tilde\alpha, \tilde\beta$ are horizontal lifts, and $g^{-1} = \mathrm{Hol}(\alpha)$ and $h^{-1} = \mathrm{Hol}(\beta)$.

Now since $\mathcal{H}$ is G-invariant, $\tilde\beta \cdot g$ is horizontal with initial point $p \cdot g$, and terminal point $(p\cdot g)\cdot h$. So as our horizontal lift of the product $\alpha\beta$ take $\tilde\alpha \cdot (\tilde\beta \cdot g)$ and compare endpoints:

Terminal point is $(p \cdot h)\cdot g = p\cdot(hg)$ and so $\mathrm{Hol}(\alpha\beta) = (hg)^{-1} = g^{-1}h^{-1}$ and $\mathrm{Hol}(\alpha)\mathrm{Hol}(\beta) = g^{-1}h^{-1}$.

Similarly $\mathrm{Hol}_{p\cdot g} = g^{-1}\mathrm{Hol}_p g$. The point:

The Hol map descends to

$$\{(E, \mathcal{H}) \text{ flat } G\text{-bundles}\}\Big/\sim \longrightarrow \mathrm{Hom}(\pi_1(B), G)\Big/G$$

(up to equiv).

In fact, it's a bijection. The inverse of this map is given by: $\rho : \pi_1(B) \longrightarrow G$ maps to $\tilde B \times G \big/ \sim$, where the action on $\tilde B$ is the natural one and on G it's ρ.