Introduction to Hyperbolic Metric Spaces

Adriana-Stefania Ciupeanu

University of Manitoba
Department of Mathematics

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UNIVERSITY OF MANITOBA

Introduction

Geometric group theory is relatively new, and became a clearly identifiable branch of mathematics in the 1990s due to Mikhail Gromov.

Hyperbolicity is a centre theme and continues to drive current research in the field.

Geometric group theory is bases on the principle that if a group acts as symmetries of some geometric object, then we can use geometry to understand the group.

Introduction

Gromov's notion of hyperbolic spaces and hyperbolic groups have been studied extensively since that time.

Many well-known groups, such as mapping class groups and fundamental groups of surfaces with cusps, do not meet Gromov's criteria, but nonetheless display some hyperbolic behaviour.

In recent years, there has been interest in capturing and using this hyperbolic behaviour wherever and however it occurs.

If Euclidean geometry describes objects in a flat world or a plane, and spherical geometry describes objects on the sphere, what world does hyperbolic geometry describe?

Hyperbolic geometry takes place on a curved two dimensional surface called hyperbolic space.

The essential properties of the hyperbolic plane are abstracted to obtain the notion of a hyperbolic metric space, which is due to Gromov.

Hyperbolic geometry is a non-Euclidean geometry, where the parallel postulate of Euclidean geometry is replaced with:

For any given line R and point P not on R, in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R.

Because models of hyperbolic space are big (not to mention infinite), we will do all of our work with a map of hyperbolic space called the Poincaré disk.

The Poincaré disk

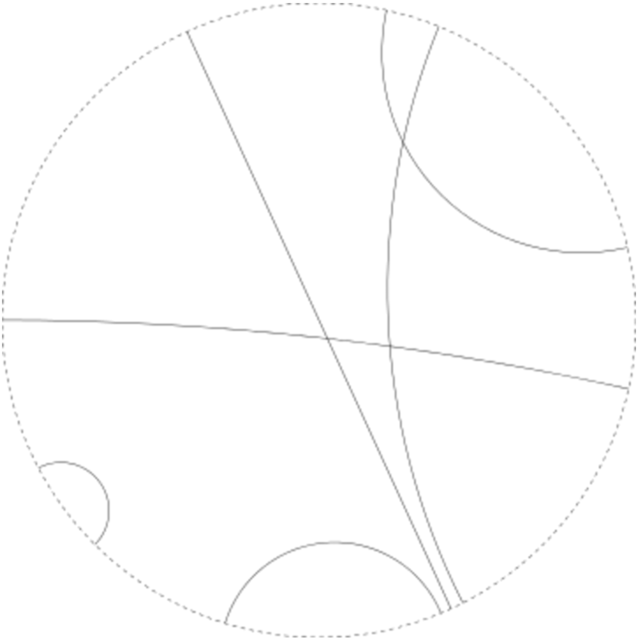
$$\mathbb{D} = \{(x, y) \mid x^2 + y^2 < 1\}$$

is the inside of a circle (although the circle is not included) with metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

In the Poincaré disk model, geodesics appear curved. They are arcs of circles. Specifically:

- Geodesics are arcs of circles which meet the edge of the disk at $\frac{\pi}{2}$.
- Geodesics which pass through the center of the disk appear straight.



The essential properties of the hyperbolic plane are abstracted to obtain the notion of a hyperbolic metric space, which is due to Gromov.

Geodesic Metric Spaces

Definition

Let (X, d) be a metric space and $p : [0, 1] \to X$ be a path. If the supremum over all finite partitions $[t_0 = 0,\ t_2,\ \ldots,\ t_{n-1},\ t_n = 1]$ of

$$\sum_{i=1}^{n} d(p(t_{i-1},\ p(t_1)),$$

*exists then we say that p is a **rectifiable** path. We denote the supremum by $l(p)$ and call it the **length** of p. Furthermore, $(X,\ d)$ is a **path metric space** if for all x_1 and x_2 in X, we have that*

$$d(x_1,\ x_2) = \inf\{l(p) \,|\, p \text{ is a rectifiable path in } X \text{ from } x_1 \text{ to } x_2\}.$$

Geodesic Metric Spaces

The idea is as follows. Suppose you want to formally define the notion of length of a path. One property you would like is that if you approximate your curve by a concatenation of “straight lines” then the length of such concatenation approximates the length of the path (well, if the path has finite length). The length of a “straight line” should be the same as the distance between the endpoint.

Definition

A metric space X is geodesic if between every two points of X there exists a geodesic.

Geodesic Metric Spaces

A geodesic metric space is a length space in which infima of distances of rectifiable paths are attained. Such a space clearly has to be path connected.

Example

A connected graph is a geodesic metric space. We give each edge length and as above the distance between two vertices v_1 and v_2 is the least number of edges of a path between v_1 and v_2.

Example

Example

Consider the unit circle S^1 in the plane. There are two natural metrics we could put on S^1:

- The first is the induced Euclidean metric: the distance between two points is the length of the straight line in $\mathbb{R}^2$ between them.
- The other is the arc length metric: the distance between two points is the length of the (shortest) circular arc between them.

The first of these is not a geodesic metric (since, for example, there is no path is S^1 of length 2 connecting a pair of antipodal points), whereas the second one is geodesic.

Slim and Thin Triangles

We want to know what it means for a space (X, d) to be hyperbolic. There are several definitions, all of them being equivalent.

We will give various versions of Gromov's hyperbolic criterion.

Definition

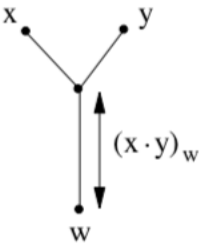
*Suppose that (X, d) is a metric space with basepoint w. Then we define the **Gromov product** on $X \times X$ based at w by*

$$(x \cdot y)_w = \frac{1}{2}\left(d(x, w) + d(y, w) - d(x, y)\right).$$

This is sometimes called an inner product. However, it is not an inner product in the usual sense as it is not necessary defines on a vector space.

Example

Suppose that x, y, and w are points in a tree T, let γ be a geodesic from w to x and let η be a geodesic from w to y. Then $(x \cdot y)_w$ is the distance along γ (or η) before the two geodesics separate.

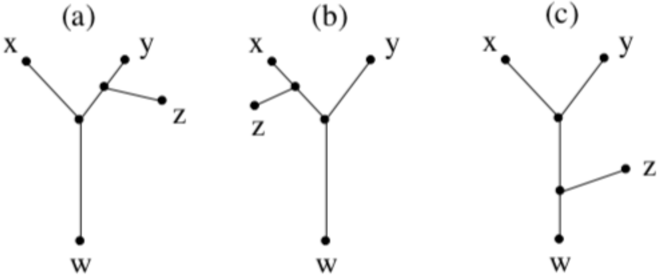


Example

Choose any basepoint w in T. Then the metric d on T satisfies the property that for all x, y, and z in T we have that

$$(x \cdot z)_w \geq \min\{(x \cdot y)_w, (y \cdot z)_w\}.$$

We have that there are three cases:



Example

In case (a) we have

$$(x \cdot z)_w = (x \cdot y)_w = \min\{(x \cdot y)_w, (y \cdot z)_w\},$$

in (b) we have

$$(x \cdot z)_w > (x \cdot y)_w = (y \cdot z)_w,$$

and in (c) we have

$$(x \cdot z)_w = (y \cdot z)_w = \min\{(x \cdot y)_w, (y \cdot z)_w\}.$$

Slim and Thin Triangles

If we relax the property satisfies by the metric on a tree in the above example then we get the following:

Definition

If (X, d) is a metric space with basepoint w and there exists $\delta \geq 0$ such thta for all x, y, and z in X we have

$$(x \cdot z)_w \geq \min\{(x \cdot y)_w, (y, z)_w\} - \delta,$$

then we say that the Gromov product based at w is δ-hyperbolic. If there exists $\delta \geq 0$ such that the Gromov product is δ-hyperbolic, we just say that the Gromov product based at w is hyperbolic.

Slim and Thin Triangles

We next consider geodesic triangles in a geodesic metric space, i.e. triples of points with a specified geodesic (called a side) between every pair of points. If (x, y, z) is such a triangle then we denote the geodesic between, say, x and y by $[x, y]$.

Definition

Let (X, d) be a geodesic metric space and let $\triangle$ be a geodesic triangle. We say that $\triangle$ is δ-slim if for each ordering (A, B, C) of its sides, and for any point $w \in A$, $w \in B$, $w \in B$ we have

$$min\{(w, B), d(w, C)\} \leq \delta$$

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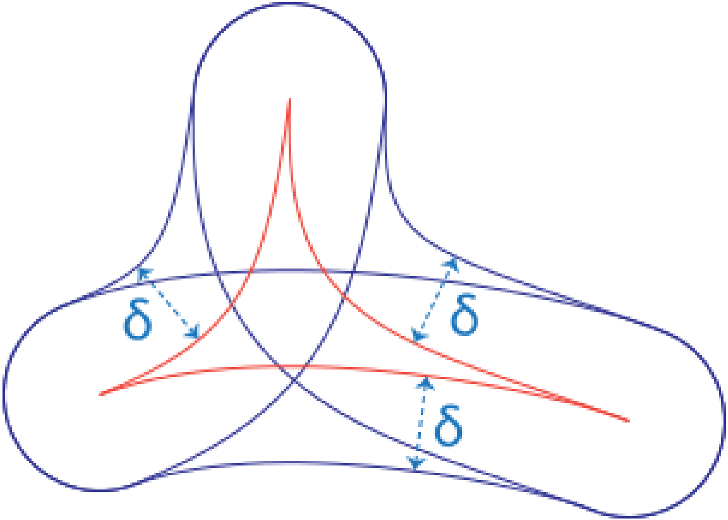
Slim and Thin Triangles



Slim and Thin Triangles

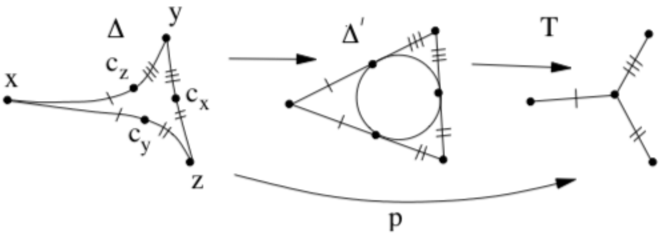
In other words, we have that a $\delta-$ neighbourhood of any two sides contains the third side.

Typical euclidean triangles are not all δ-slim for some fixed δ. If you fix δ, just choose a very large triangle.

Definition

Let X be a geodesic metric space. Given a geodesic triangle δ in X, we consider the Euclidean triangle $\triangle'$ with the same side lengths. Collapse $\triangle'$ to a tripod T. Let $p : \triangle \to T$ be the map which so arises. We say that $\triangle$ is δ-thin if for all $t \in T$ we have $diam(p^{-1}(t)) \leq \delta$.

Slim and Thin Triangles



Hyperbolic Metric Space

Proposition

Let X be a geodesic metric space. Then the following are equivalent.

1. *There exists a point $w \in X$ such that the Gromov product based at w is hyperbolic.*
2. *For all points $w \in X$ the Gromov product based at w is hyperbolic.*
3. *There exists $\delta \geq 0$ such that every geodesic triangle in X is δ-slim.*
4. *There exists $\delta \geq 0$ such that every geodesic triangle in X is δ-thin.*

Definition

If a geodesic metric space satisfies any of the properties in the last proposition we call it a ***hyperbolic metric space****.*

Hyperbolic Metric Spaces

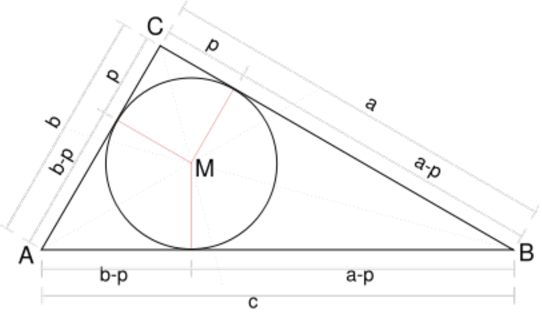
Examples

Any tree is 0-hyperbolic. Every geodesic triangle is a “tripod“.

Example

$\mathbb{R}^2$ is not δ-hyperbolic for any δ.

Example Hyperbolic Plane



Example Hyperbolic Plane

The hyperbolic plane is hyperbolic.

- Incircle of a geodesic triangle is the circle of largest diameter contained in the triangle
- Every geodesic triangle lies in the interior of an ideal triangle, all of which are isometric with incircles of diameter 2 log 3.
- The Gromov product also has a simple interpretation in terms of the incircle of a geodesic triangle.
- The quantity $(A, B)_C$ is just the hyperbolic distance p from C to either of the points of contact of the incircle with the adjacent sides: $c = (a - p) + (b - p)$, so that $p = \dfrac{(a + b - c)}{2} = (A, B)_C$.

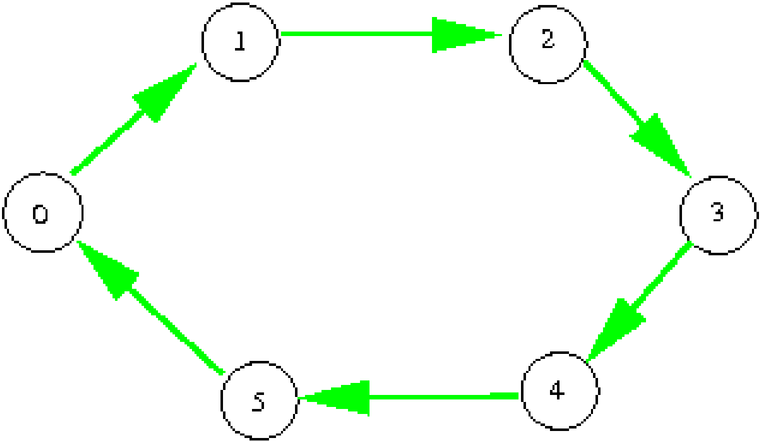
Cayley Graph

It can be showed that if two spaces are quasi-isomteric and one of them is hyperbolic, then so is the other (the constant δ might change).

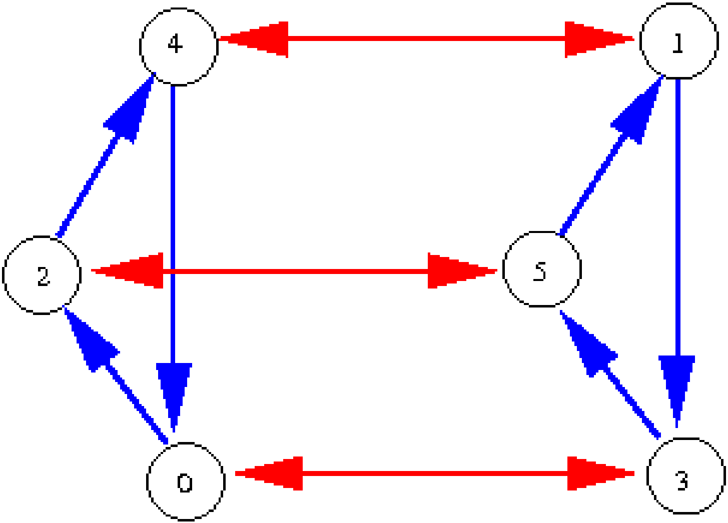
In particular, we have that a finitely generated group G is hyperbolic if and only the Cayley graph of G is hyperbolic. However, this is up to quasi-isometry.

Furthermore, we have that given a group G and different generating sets the Cayles graphs are different.

Cayley Graph of $\mathbb{Z}/6\mathbb{Z}$ with generating set $\{1\}$



Cayley Graph of $\mathbb{Z}/6\mathbb{Z}$ with generating set $\{2, 3\}$



Quasi-Isometries

Definition

Let (X, d_X) *and* (Y, d_Y) *be two metric spaces. A map* $f : X \to Y$ *is a* (λ, c) *QI embedding if for all* $x, y \in X$ *we have (*$\lambda \geq 1, c \geq 0$*)*

$$\frac{d_X(x,y)}{\lambda} - c \leq d_Y(f(x), f(y)) \leq \lambda d_x(x,y) + c.$$

Furthermore, the (λ, c) *QI embedding is a* (λ, c) *QI is for every* $y \in Y$ *there exists* $x \in X$ *where*

$$d_y f(x), y) \leq c.$$

Quasi-Isometries

Example

If X and Y are both bounded metric spaces, then X and Y are quasi-isometric.

Example

Any finite radius metric space (X, d) is QI to a point. Let $\lambda = \sup\{d(x, y) \mid x, y \in X\}$. Let $c = 1$. Furthermore, let $f : X \to \{p\}$ be the constant map. We want to show that for every $x, y \in X$

$$\frac{d(x, y)}{\lambda} - c1 \leq (f(x), f(y)) = \leq \lambda d(x, y) + 1.$$

Since, is onto it is QI.

Quasi-Isometry

Example

For $v, b \in R^2$, the map $t \mapsto tv + b$ from $\mathbb{R}$ to $\mathbb{R}^2$ is a quasi-isometric embedding.

Example

The map $t \mapsto t^2$ from $\mathbb{R}$ to $\mathbb{R}$ is not a quasi-isometric embedding. The second inequality is the one that fails.

Example

The map $t \mapsto \sqrt{t}$ from $\mathbb{R}$ to $\mathbb{R}$ is not a quasi-isometric embedding. This time the first inequality fails.

Quasi-Isometry

Example

Let G be a group given as the quotient $\pi : F(S) \to G$ and also given as $\pi' : F(S') \to G$, where S and S' are both finite sets. Now, suppose that d and d' are corresponding on G with respect to S and S'. Now, we have

$$1 : G \to G$$

is a QI between (G, d) to (G, d')
Let $\lambda_1 = \max\{d'(\pi(s), 1) \mid s \in S\}$, $\lambda_2 = \max\{d'(\pi'(s'), 1) \mid s' \in S'\}$ and $\lambda = \max\{\lambda_1, \lambda_2\}$. Then for all $x, y \in G$ we have

$$\frac{d_S(x, y)}{\lambda} \leq d_{S'}(x, y) \leq \lambda d_S(x, y)$$

Quasi-Isometry

Example (Continuation)

Let us prove that

$$\frac{d_S(x,y)}{\lambda} \le d_{S'}(x,y).$$

Suppose that $d_{S'}(x,y) = n$, for some $n \in \mathbb{Z}$, then there exists $s'_1, \dots, s'_n$ such that

$$\pi(s'_1, \dots, s'_n) = x^{-1}y \implies \pi(s'_1) \dots \pi(s'_n) = x^{-1}y.$$

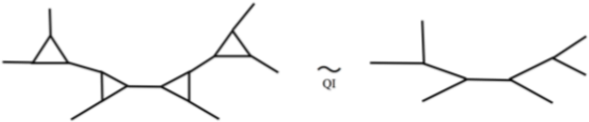
This will give us a walk of length λn between 1 to $x^1 y$ in G considered as the quotient of S. Therefore, $d_S(x,y) \le \lambda n$. Hence, we have that

$$\frac{d_S(x,y)}{\lambda} \le d_{S'}(x,y).$$

Quasi-Isometry

Example

Consider the graph in the figure below. Collapsing each of the triangles to a point gives a quasi-isometry of this graph onto a tree.



Quasi-Isometry

Definition (Quasi-inverse)

Let $f : (X, d) \to (Y, d')$ be a map of metric spaces. We say that $g : (Y, d') \to (X, d)$ is a quasi-inverse of if there exists some $D \geq 0$ such that for all $x \in X$ we have $d(g \circ f(x), x) \leq D$ and for all $y \in Y$ e have $d(f \circ g(y), y) \leq D$.

Proposition

Composition of two compatible quasi-isometries is again a quasi-isometry.

Conclusion

Suppose that we have proved that a group G is hyperbolic. Then what?

- G is finitely generated.
- For $g \in G$ the centralizer $C_G(g)$ is virtually cyclic, i.e., contains a finite index cyclic group.
- G has solvable word problem, i.e., if we are given an arbitrary word in the generators of G then there exists an algorithm to decide whether or not this word represents the identity.
- Same for the conjugacy and isomorphism problems.