Free subgroups of groups of homeomorphisms.

This talk expands upon a remark I made in my last talk:

I proved that $F_n < \mathrm{Homeo}_+(\mathbb{R})$ by exhibiting an explicit set of free generators using the ping-pong lemma. Then I said:

"Generically, any homeomorphisms will generate a free group"

<u>What did this mean?</u>

Recall that if X is a top. space, then $A \subset X$ is nowhere dense if $\mathrm{int}(\bar{A}) = \emptyset$. A sets A is called <u>first category</u> if it is a countable union of nowhere dense sets, and second category otherwise.

<u>E.g.</u> • $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is first category since $\mathrm{int}\overline{\{q\}} = \emptyset$.

• $\mathbb{R} \setminus \mathbb{Q}$ is second category.

In general, first category sets are "sparse" and second category are "abundant".

<u>Theorem</u>: (Baire) Every complete metric space is of the second category.

(In particular, in any such space the complement of first category set is of second category).

In our situation: Let $C(X,Y)$ denote the space of continuous maps $X \longrightarrow Y$ topologized with the compact-open topology:

For each pair (K,O) with $K \subseteq X$ compact and $O \subseteq Y$ open, set

$$[K,O] = \{ f : X \longrightarrow Y \mid f \text{ is continuous and } f(K) \subseteq O \}$$

These sets are a subbasis for the topology on $C(X,Y)$.

Theorem: The compact open topology on $C(X,Y)$ is completely metrizable if and only if Y is completely metrizable and X is hemicompact.

Def: A space X is hemicompact if it admits a sequence $\{K_i\}_{i \in \mathbb{N}}$ of compact subsets such that every compact $K \subset X$ lies inside some K_i.

E.g.
- If X is compact, it's hemicompact (set $K_i = X$ $\forall i$)
- $\mathbb{R}$ is hemicompact (set $K_i = [-i, i]$).

Now consider $\mathrm{Homeo}_+(\mathbb{R})$ or $\mathrm{Homeo}_+([0,1])$. Since $\mathbb{R}$ is completely metrizable and hemicompact, so is $C(\mathbb{R},\mathbb{R})$. Then

$$\mathrm{Homeo}_+([0,1]) \subseteq \mathrm{Homeo}_+(\mathbb{R}) \subset C(\mathbb{R},\mathbb{R})$$

so each of $\mathrm{Homeo}_+([0,1])$ and $\mathrm{Homeo}_+(\mathbb{R})$ is metrizable. It's not hard to see that each is also complete (e.g. a sequence $\{f_n\}$ converging to f in the compact open topology on $C(\mathbb{R},\mathbb{R})$ will yield a cts inverse if each f_n has a continuous inverse)

In our case:

Set $Y = \mathrm{Homeo}_+([0,1]) \times \mathrm{Homeo}_+([0,1])$, it is a complete metric space. Let $F_2 = F(a,b)$, and given $w \in F_2$ and $(f,g) \in Y$ write $w(f,g)$ for the function obtained by replacing:

$$a^{\pm 1} \longmapsto f^{\pm 1}$$
$$b^{\pm 1} \longmapsto g^{\pm 1}.$$

E.g. $w = aba^{-1}b^{-1} \Rightarrow w(f,g) = fgf^{-1}g^{-1}$.

For $w \neq id$, set

$$Y_w = \{(f,g) \mid w(f,g)(x) = x \text{ for all } x \in [0,1]\}, \text{ it is closed.}$$

(Ghys' exposition)

<u>Theorem</u>: For all $w \in F_2$, $\mathrm{int}(Y_w) = \emptyset$. In particular,

$$\bigcup_{w \in F_2} Y_w = \{(f,g) \mid \exists w \text{ such that } w(f,g)(x) = x \;\; \forall x \in [0,1]\}$$

is a ~~category~~ first category set consisting of exactly the pairs $(f,g) \in Y$ for which $\langle f,g \rangle \subset \mathrm{Homeo}_+([0,1])$ is <u>not</u> a free group. Thus the pairs $(f,g) \in Y$ that <u>do</u> generate a free group is a set of the second category, since Y is a complete metric space.

<u>Proof</u>: Consider

$$X = Y \times [0,1], \text{ and set}$$

$$X_w = \{(f,g,x) \mid w(f,g)(x) = x\} \supset Y_w \times [0,1].$$

Then X_w's are closed, $\boxed{\text{and } \operatorname{int}(X_w) = \emptyset}$ this takes work!

Moreover

$\operatorname{int}(Y_w) \times \operatorname{int}([0,1]) = \operatorname{int}(Y_w \times [0,1]) \subset \operatorname{int}(X_w) = \emptyset,$
meaning $\operatorname{int}(Y_w) = \emptyset$. So all that's missing is $\operatorname{int}(X_w) = \emptyset$.

<u>Idea</u>: Choose w of minimal length such that $\exists\, U$ open in $\operatorname{int}(X_w)$. Use minimality to show that for an appropriately chosen $(f,g,x) \in X_w$ we can perturb very slightly to produce $(\bar{f},\bar{g},x) \in U$ with $w(\bar{f},\bar{g})(x) \neq x$, a contradiction. (Ghys).

In general, there's nothing special about $[0,1]$.

<u>Theorem</u>: If M is any manifold, then any generic pair (f,g) of elements in $\mathrm{Homeo}(M)$ will generate a free group.

Contrast this with the following:

Let $PL_+([0,1])$ denote the piecewise-linear homeomorphisms of $[0,1]$ preserving order, ie $f : [0,1] \longrightarrow [0,1]$ with $f(0) = 0$, $f(1) = 1$ and f is affine on the complement of some finite set.

Theorem (Brin and Squier, 1985).

The group $PL_+([0,1])$ does not contain a nonabelian free group.

Proof: (Sketch) (Ghys)

If $f \in PL_+([0,1])$, write $\operatorname{supp}_0(f)$ to mean the set of non-fixed points of f ("open support").

Suppose $f, g \in PL_+([0,1])$ generate a free group. Then $I = \operatorname{supp}_0(f) \cup \operatorname{supp}_0(g)$ is a union of finitely many open intervals $I_1, \ldots, I_n$. $\boxed{\text{Trick}}$:

Note that

$$\overline{\operatorname{supp}_0(fgf^{-1}g^{-1})} \subset \operatorname{supp}_0(f) \cup \operatorname{supp}_0(g)$$

since near the boundary of I both f ~~ang~~ and g are linear, and thus commute. So

$$S = \{ h \in F_2 \mid \overline{\operatorname{supp}_0(h)} \subset \operatorname{supp}_0(f) \cup \operatorname{supp}_0(g) \}$$

is nonempty, and we can choose an element $y \in S$ such that $\operatorname{supp}_0(y)$ ~~$\cap I$ has~~ meets the fewest intervals $I_1, \ldots, I_n$. Using minimality, you can cook up $k \in F_2$ such that $ky = yk$, a contradiction.

Remark: Unlike before, this does not generalize to all other manifolds - not even compact ones, or some other nice class. E.g. $PL_+(S^1)$ contains plenty of free subgroups. In fact:

<u>Corollary of Margulis' Theorem</u> (2000).

Let $\Gamma \subset \mathrm{Homeo}_+(S^1)$ be such that all orbits are dense in the circle. Then exactly one of the following holds:

(i) Γ contains a nonabelian free group

(ii) Γ is abelian and conjugate to a group of rotations.

So, for example:

Take a subgroup $A \subset \mathrm{Homeo}_+(S^1)$ of irrational rotations. All orbits are dense, by irrationality of the rotation — but it satisfies (ii), so no free groups are contained in A.

Choose any $f \in PL_+(S^1)$ that does not commute with all elements of A. Then

$$\langle A, f \rangle \subset PL_+(S^1)$$

satisfies (i) above. Note that there are <u>many</u> possible choices of f, so that there are many free groups in $PL_+(S^1)$.