Teichmüller space of compact surfaces.

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(1) Riemann Surfaces

Definitions: A Riemann surface R is a topological space which is Hausdorff and second countable which is locally homeomorphic to $\mathbb{R}^2$ with an atlas of charts $\phi: U \longrightarrow \mathbb{C}$ (where U is open in R)

meaning:
- All ϕ's are homeomorphisms onto their images
- Given two charts $\phi: U \to \mathbb{C}$, $\psi: V \to \mathbb{C}$ the map $\phi \circ \psi^{-1}: \psi(U \cap V) \longrightarrow \phi(U \cap V)$ is a biholomorphism.

(This is like a differentiable manifold with biholomorphic transition functions).

Ex: ① Let G be the group action $z \mapsto z + 2\pi i n$ for $n \in \mathbb{Z}$. Then $\mathbb{C}/G$ is a Riemann surface homeomorphic to the cylinder.

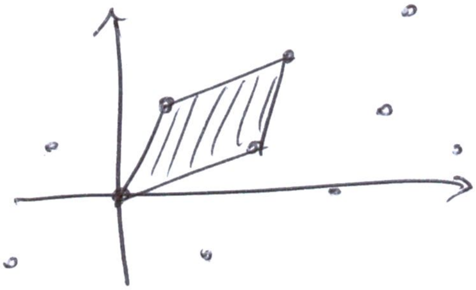
② Ω an open, connected subset of $\mathbb{C}$ (just use a single chart for the whole Riemann surface)

③ Let Γ be the lattice $\text{span}_{\mathbb{Z}}\{w_1, w_2\}$ where $w_1, w_2 \in \mathbb{C}$ are nonzero and linearly independent over $\mathbb{R}$.

We say $z \sim w$ if $z - w \in \Gamma$.

Then $\mathbb{C}$ has a tiling by parallelograms, and we can identify

$\mathbb{C}/\Gamma = \mathbb{C}/\sim$ with a fundamental domain



, this gives $\mathbb{C}/\Gamma \sim$

Then upon making charts for $\mathbb{C}/\Gamma$, we can (upon choosing charts appropriately) ensure that our transition functions are translations by elements of Γ.

This last example (in fact all 3) are a bit deceptive in that they're covered by the plane. This is not usual, in that "most" surfaces are $\mathbb{D} = \{ z \mid |z| < 1 \}/G$ for some G.

Riemann moduli space

Like isomorphism in group theory and homeomorphism in topology, we need an appropriate equivalence of Riemann surfaces.

Ex: $\mathcal{M}(g) = \left\{ \begin{array}{c} \text{Riemann surfaces of genus } g \\ \text{which are compact} \end{array} \right\} / \sim$

where $R_1 \sim R_2$ if there's a biholomorphism

$\sigma : R_1 \to R_2$ between them.

Example: A non-compact example.

$$\mathcal{M}(A) = \left\{ \begin{matrix} \text{2-connected domains in} \\ \text{the plane} \end{matrix} \right\} / \sim$$

(again, $\sim$ is biholomorphic equivalence)
By 2-connected, we essentially mean that the fundamental group is $\mathbb{Z}$.
Can show: Every 2-connected Riemann surface is $\mathbb{C} \setminus \{0\}$ or $\{ z \mid 1 < |z| < R \}$ for $R \in (1, \infty]$ up to biholomorphism.

"Theorem" (Riemann) $\mathcal{M}(g)$ is a complex manifold of dimension $3g-3$ in the case that $2g-2 > 0$. (So not true for the torus or sphere).

This last theorem is not quite true, as it's an orbifold you get. A rigorous version of this theorem would say that the Teichmüller space is dimension $3g-3$. So what is Teichmüller space?

Ex: Consider tori $\mathbb{C}/_{\{w_1, w_2\}}$ as above. When are two such tori equivalent?

Answer: $\mathbb{C}/_{\Gamma} \sim \mathbb{C}/_{\Gamma'} \iff \exists\, \nu \in \mathbb{C} \setminus \{0\}$

such that $\nu\Gamma = \Gamma'$.

Proof: Let $f: \mathbb{C}/\Gamma \longrightarrow \mathbb{C}/\Gamma'$ be a biholomorphism. If so, $\exists$ a biholomorphism $\mathbb{C} \xrightarrow{g} \mathbb{C}$ such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "g"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Gamma \arrow[r, "f"'] & \mathbb{C}/\Gamma' \end{tikzcd}$$

commutes (ie, we need to know an existence of lifts)

But a biholomorphism $g: \mathbb{C} \longrightarrow \mathbb{C}$ must be of the form $g(z) = \nu z + \mu$ for $\nu, \mu \in \mathbb{C}$. But if $g: \mathbb{C} \longrightarrow \mathbb{C}$ is to descend to a map f as above, then $\nu z_1 + \mu \sim \nu z_2 + \mu$ whenever $z_2 - z_1 \in \Gamma$. So $\nu(z_2 - z_1) \in \Gamma'$ whenever $z_2 - z_1 \in \Gamma$. So $\nu\Gamma \subset \Gamma'$! Apply the same argument to g^{-1} to get $\frac{1}{\nu}\Gamma' \subset \Gamma$.

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(2) Definition of Teichmüller space (for compact surfaces)

Definition: A marking of a Riemann surface of genus g is a homeomorphism of the surface S with S_0, ie $g: S_0 \longrightarrow S$ where S_0 is fixed, modulo isotopy.

ie. Fix S_0, a Riemann surface of genus g

We say $g_1 : S_0 \longrightarrow S$ is equivalent to $g_2 : S_0 \longrightarrow S$ if $g_2^{-1} \circ g_1$ is homotopic to the identity.

<u>Definition</u>: The Teichmüller space $T(g)$ of Riemann surfaces is, having fixed S_0,

$$T(g) = T_{S_0}(g) = \{(S, h)\}/\sim$$

where S is a Riemann surface, $h : S_0 \longrightarrow S$ is a marking and $(S_1, h_1) \sim (S_2, h_2)$ if there's a biholomorphism $\sigma : S_1 \longrightarrow S_2$ such that $\sigma \circ h_1$ is homotopic to h_2.

(<u>Remark</u>: This is like Riemann equivalence that preserves a marking, so $T(g)$ is bigger than $\mathcal{M}(g)$.)

<u>Definition</u>: The <u>modular group</u> $\mathrm{Mod}(g)$ is $\{g : S_0 \longrightarrow S_0\}/\sim$ where g are homeomorphisms and $\sim$ is up to homotopy.

Then $\mathrm{Mod}(g)$ acts on $T(g)$ by

$$[g] \cdot [S, h] = [S, h \circ g^{-1}].$$

The idea is that

(i) $\mathcal{M}(g) = T(g)/\mathrm{Mod}(g)$

(ii) $T(g)$ is a complex manifold of dim $3g-3$ if $2g-2>0$. [Teichmüller, Ahlfors/Bers]

Ex: $T(1)$. (the torus).

Let $\Gamma = \text{span}_{\mathbb{Z}}\{w_1, w_2\}$.

Theorem: $\Gamma = \Gamma'$ iff $\exists M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = \pm 1$ with $a, b, c, d \in \mathbb{Z}$ and $\begin{pmatrix} w_2' \\ w_1' \end{pmatrix} = M \begin{pmatrix} w_2 \\ w_1 \end{pmatrix}$.

Proof: (sketch). We must be able to write

$$w_2' = a w_1 + b w_2$$
$$w_1' = c w_1 + d w_2$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. This determines M.

Now returning to our discussion of $T(1)$.

We can always assume $w_1 = 1$ and $w_2 \in \mathbb{H} = \{ z \mid \text{Im}(z) > 0 \}$, by rescaling and re-ordering.

Theorem: $\mathbb{C}/\langle 1, \tau \rangle \sim \mathbb{C}/\langle 1, \tau' \rangle$ (Riemann equivalence)

if and only if $\tau' = T(\tau)$ for some Möbius transform

$$T(w) = \frac{aw + b}{cw + d}$$ where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

Punch line: $T(1) = \mathbb{H}$. Fixing one generator as 1, we imagine varying our other lattice generator over $\mathbb{H}$. Here, $\text{Mod}(1) = PSL(2, \mathbb{Z})$ and

$$\mathcal{M}(1) = \mathbb{H}/PSL(2, \mathbb{Z}).$$