Learning seminar – 08 Oct 2015

Vector bundles

This talk is based off material in Milnor–Stasheff's book Characteristic Classes.

Let B be a top. space.

A vector bundle over B consists of

1) cont's map $\pi : E \longrightarrow B$, such that
2) each fibre $\pi^{-1}(b)$ has a vector space structure

satisfying a local triviality condition:

$\forall\, b \in B$, $\exists$ nbhd U, $n \in \mathbb{Z}_{\geq 0}$ and homeo $h : U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$

so that $x \longmapsto h(b,x)$ is isomorphism $\mathbb{R}^n \xrightarrow{\cong} \pi^{-1}(b)$.

in particular $h(b,x) \in \pi^{-1}(b)$.

From now on, assume spaces are connected, unless stated otherwise.

Examples

0) trivial bundle $E = B \times \mathbb{R}^n$, $\pi = pr_1$.

1) If $B = M$ is a manifold, $E = TM$ (tangent space) w/ natural map $\pi : TM \longrightarrow M$

Vector bundles $\pi : E \longrightarrow B$ and $\pi' : E' \longrightarrow B$ are isomorphic if $\exists$ homeo $f : E \longrightarrow E'$ that restricts to an isomorphism on each fibre.

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \cup & & \cup \\ \pi^{-1}(b) & \xrightarrow{\cong} & (\pi')^{-1}(b) \end{array}$$

When $TM \simeq M \times \mathbb{R}^n$, we say M is parallelizable.

Recall that a vector field is a (smooth) assignment

$$M \ni x \longmapsto (x,v) \in \underset{\underset{\pi^{-1}(x)}{\shortparallel}}{T_x M} \subset TM$$

That is, vector fields are (cross)-sections $s : M \longrightarrow TM$.

(Generally, $s : B \to E$ is a section if $\pi \circ s = id_B$. Note all vector bundles have a section — zero section.)

So, M is parallelizable $\iff \exists\ n = \dim M$ vector fields that are nowhere dependent (ie lin. ind. at each $x \in M$)

Ex · $M = S^2$ is not parallelizable (e.g. by hairy ball thm)

· $M = S^3$ is; so is any compact orientable 3-manifold (related to "spin" structure)

· S^n parallelizable $\iff n = 0, 1, 3, 7$. (Milnor, Bott, Kervaire, ...)

Generally, a $\mathbb{R}^n$-bundle $\pi: E \to B$ is trivial $\iff \exists\ n$ nowhere dependent cross sections $s_i : B \to E$.

Recall $\mathbb{RP}^n = \{1\text{-dim subspaces of } \mathbb{R}^{n+1}\} = S^n/_{x \sim (-x)}$. (denote $[x] = \{x, -x\}$)

There is a natural vector bundle over $B = \mathbb{RP}^n$, denoted γ_1^n:

$$\{\text{line thru } x, -x\} = \pi^{-1}(x) \subset E(\gamma_1^n) = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in \operatorname{span}\{x\}\}$$

$$\begin{array}{ccc} E(\gamma_1^n) & & ([x],v) \\ \downarrow & & \downarrow \\ \mathbb{RP}^n & & [x] \end{array}$$

(an $\mathbb{R}^1$-bundle)

For $n = 1$, $E(\gamma_1^1)$ is familiar:

We can write pts $e \in$ as $e = ([(\cos\theta, \sin\theta)], t(\cos\theta, \sin\theta))$, $t \in \mathbb{R}$, $0 \le \theta \le \pi$

This description is unique, unless $\theta = 0, \pi$:

$$\underbrace{([(1,0)], t(1,0))}_{\theta = 0} = \underbrace{([(-1,0)], -t(-1,0))}_{\theta = \pi}$$

That is $E(\gamma_1^1) \cong [0,\pi] \times \mathbb{R} /_{(0,t) \sim (\pi, -t)}$ = (open) Möbius band.

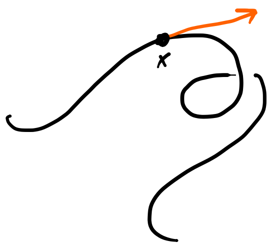
In particular, γ_1^1 is not trivial.

Prop γ_1^n is not trivial $\forall\ n \ge 1$.

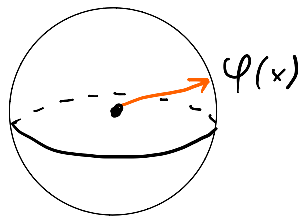
proof: It suffices to show $\nexists$ a non-zero section $s: \mathbb{RP}^n \to E(\gamma_1^n)$. Suppose not, and consider $S^n \to \mathbb{RP}^n \to E(\gamma_1^n)$, sending $x \mapsto ([x], t(x)x)$. $t: S^n \to \mathbb{R}$ must satisfy $t(-x) = -t(x)$ (as $([-x], t(-x)(-x)) = ([x], t(x)x)$). But IVT $\Rightarrow \exists\ x_0$ s.t. $t(x_0) = 0$ so that $s([x_0])$ vanishes. ∎

Grassmannian manifolds

Recall Gauss map for (oriented) 1-manifolds $M^1 \subset \mathbb{R}^3$:



$x \longmapsto \varphi(x) \in S^2$
unit tangent vector



$$\varphi : M^1 \longrightarrow S^2$$

and also for (oriented) hypersurface $M^n \subset \mathbb{R}^{n+1}$, $\varphi : M^n \to S^n$.

$\varphi(x)$ describes tangent space (or normal space, equivalently)

For submanifold $M^n \subset \mathbb{R}^{n+k}$, we'll view assignment $x \longmapsto T_xM$ as a map into some space.

- Stiefel manifold $V_n(\mathbb{R}^{n+k}) = \{ \text{n-frames in } \mathbb{R}^{n+k} \}$

  n-tuple of lin. indep vectors in $\mathbb{R}^{n+k}$

  ↳ an open subset of $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$ ⇒ it's a manifold.

  ↳ can view as a 'homogeneous space' $V_n(\mathbb{R}^{n+k}) = GL_{n+k}(\mathbb{R}) / GL_k(\mathbb{R})$

- $G_n(\mathbb{R}^{n+k}) = \{ \text{n-dim subspaces of } \mathbb{R}^{n+k} \}$ topologized as quotient of $V_n(\mathbb{R}^{n+k})$ (Grassmannian mfld)

  ↳ By Gram-Schmidt, can instead view $G_n(\mathbb{R}^{n+k})$ as quotient of $V_n^{orth}(\mathbb{R}^{n+k})$, which is compact (it's closed, bounded subset of $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$)

  ⇒ $G_n(\mathbb{R}^{n+k})$ is compact

  ↳ $G_n(\mathbb{R}^{n+k})$ is a (Hausdorff) topological manifold

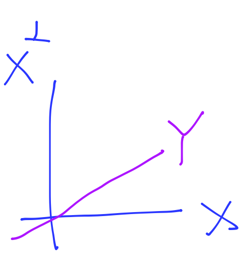
  Hausdorff: it suffices to separate $X \neq Y \in G_n(\mathbb{R}^{n+k})$ by a cont's function

  Take $w \in X$ but not in Y and consider $\mathrm{dist}(w, -) : G_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{R}$.

  locally Euclidean: For $X \in G_n(\mathbb{R}^{n+k})$, let $U = \{ Y \in G_n(\mathbb{R}^{n+k}) \mid Y \cap X^\perp = 0 \}$

  Such Y correspond to graphs of linear maps $X \longrightarrow X^\perp$

  So $U \cong \mathbb{R}^{nk}$.



Note: $G_1(\mathbb{R}^{k+1}) = \mathbb{RP}^k$.

Next we define natural vector bundles over $G_n(\mathbb{R}^{n+k})$, analogous to γ_1^n. $\leftarrow \gamma_1(\mathbb{R}^{n+1})$

Let $\gamma_n(\mathbb{R}^{n+k})$ be given as follows:

$$\begin{tikzcd} X = \pi^{-1}(X) \subset E(\gamma_n(\mathbb{R}^{n+k})) = \{(X, v) \in G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \mid v \in X\} \arrow[d, "\pi"] & (X,v) \arrow[d, mapsto] \\ G_n(\mathbb{R}^{n+k}) & X \end{tikzcd}$$

$X \cong \mathbb{R}^n \leftarrow$ $n = \dim$ of fibres

local triviality: For $X \in G_n(\mathbb{R}^{n+k})$, let $U = \{Y \in G_n(\mathbb{R}^{n+k}) \mid Y \cap X^\perp = 0\}$

$$U \times X \longrightarrow \pi^{-1}(U)$$

$$(Y, v) \longmapsto (Y, w), \text{ where } w \mapsto v \text{ via } \begin{tikzcd} Y \subset X \oplus X^\perp \arrow[d] \arrow[dr] & \\ & X \end{tikzcd}$$

Define (generalized) Gauss map for $M^n \subset \mathbb{R}^{n+k}$:

This is covered by map of vector bundles:

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & E(\gamma_n(\mathbb{R}^{n+k})) \arrow[d] \\ M \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd} \qquad \begin{aligned} (x, v) &\longmapsto (T_x M, v) \\ x &\longmapsto T_x M \subset \mathbb{R}^{n+k} \end{aligned}$$

More generally, for any $\mathbb{R}^n$-bundle (where fibers are not necessarily subspaces of $\mathbb{R}^{n+k}$) over compact Hausdorff bases B:

$\exists$ map of vector bundles for sufficiently large k.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E(\gamma_n(\mathbb{R}^{n+k})) \arrow[d] \\ B \arrow[r] & Gr_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

(Sketch of construction)

It suffices to construct $E \xrightarrow{\varphi} \mathbb{R}^m$ whose restriction to fibers is linear, injective.

Then we define $E \longrightarrow E(\gamma_n(\mathbb{R}^m))$ by $e \longmapsto (\varphi(\text{fiber thru } e), \varphi(e))$.

To that end, cover B by $U_1, \dots, U_r$ "trivializing" nbhds.

For each i, the trivialization gives

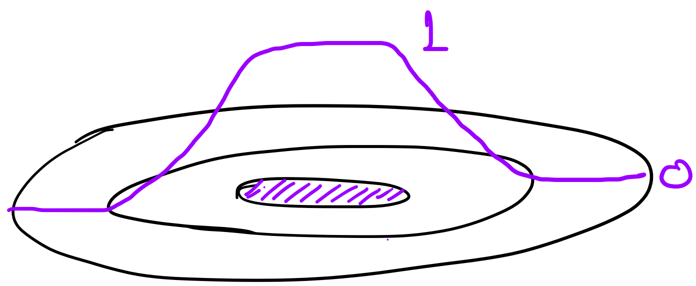
$$\varphi_i : \pi^{-1}(U_i) \longrightarrow \mathbb{R}^n$$

Idea: map each $\pi^{-1}(U_i)$ into its own summand,

$$E \longrightarrow \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n = \mathbb{R}^{nr}$$

$$e \longmapsto (\lambda_1(e)\varphi_1(e), \dots, \lambda_r(e)\varphi_r(e))$$

$0 \le \lambda_j\text{'s} \le 1$



Similar constructions hold for paracompact B, but for a bundle $E(\gamma_n(\mathbb{R}^\infty)) \longrightarrow G_n(\mathbb{R}^\infty)$.

'Universal' property

Given a vector bundle $E \xrightarrow{\pi} B$ and a map $X \xrightarrow{f} B$, can form

induced or pullback bundle: $f^*E \to X$, where

$$f^*E = \{(x,e) \mid f(x) = \pi(e)\}$$ (note fibre over $x \cong$ fibre over $f(x)$)

The existence of a map $E \longrightarrow E(\gamma_n(\mathbb{R}^{n+k}))$ induces a map on bases

$$B \longrightarrow G_n(\mathbb{R}^{n+k})$$

Prop $f^*E(\gamma_n(\mathbb{R}^{n+k}))$ is isomorphic to E (as vector bundles over B).

proof: We'll show the more general statement:

If $f: E \to E'$ is a map of vector bundles whose restriction to each fiber is an isomorphism, then $E \simeq \bar{f}^*E'$. ($\bar{f}: B \to B'$ is induced map of bases.)

But this is obvious, since there's a natural map

$$\begin{aligned} E &\longrightarrow \bar{f}^*E' \\ e &\longmapsto (\pi(e), f(e)) \end{aligned}$$

(as there is for any pullback)

that sends fibre through e to fibre over $\pi(e) \simeq$ fibre through $f(e)$. $\square$

So we have an assignment

$$\{\mathbb{R}^n\text{-bundles over } B\} \longrightarrow \{f: B \to G_n(\mathbb{R}^\infty)\}$$

and the pullback construction gives one in opposite direction.

Turns out:

$$\frac{\{\mathbb{R}^n\text{-bundles over } B\}}{\text{isomorphism}} \xleftrightarrow{1:1} \frac{\{f: B \to G_n(\mathbb{R}^\infty)\}}{\text{homotopy}}$$