Introduction to Braid Groups on Surfaces – Part II

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Summary

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Braid Groups on Surfaces – Definition

Let M be a closed connected surface, not necessarily orientable, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of M. A geometric braid over M based at $\mathcal{P}$ is an n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of paths, $\gamma_i : [0,1] \to M \times [0,1]$, such that:

(1) $\gamma_i(0) = P_i$, for all $i = 1, \ldots, n$,

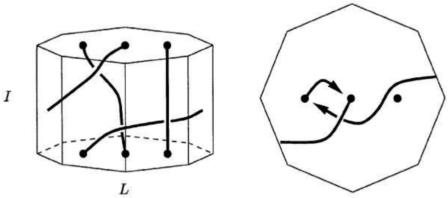
(2) $\gamma_i(1) \in \mathcal{P}$, for all $i = 1, \ldots, n$,

(3) $\{\gamma_1(t), \ldots, \gamma_n(t)\}$, are n distinct points of M, for all $t \in [0,1]$.

For all $i = 1, \ldots, n$, we will call γ_i the i-th string of γ.

Braid Diagrams

Example of a braid on 3-strands in two different views:

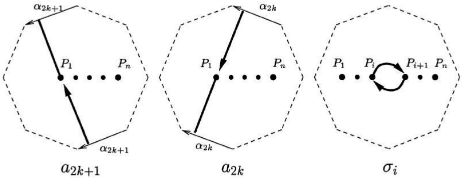


- Two geometric braids based at $\mathcal{P}$ are equivalent if there is a homotopy which deforms one of them into the other, provided that at anytime we always have a geometric braid based at $\mathcal{P}$.
- The product of two braids is induced by the usual product of paths and it endows the set of equivalence classes of braids with a group structure $B_n(M)$.
- If $\gamma_i(1) = P_i$, for all $i = 1, \ldots, n$ then we say that γ is a pure braid. It endows the group $PB_n(M)$ which is a normal subgroup of $B_n(M)$.

Theorem (Gonzalez-Meneses [GM, Theorem 2.1, p.435])

Let M be a closed, orientable surface of genus $g \geq 1$. Then $B_n(M)$, admits the following presentation:

Generators: $\{a_{1,1}, \ldots, a_{1,2g}\} \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$;



Relations:

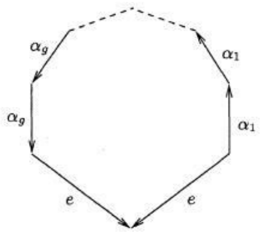
$$
\begin{array}{lll}
\text{(R1)} & \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2; \\
\text{(R2)} & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq n-2; \\
\text{(R3)} & a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1} = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1; & \\
\text{(R4)} & a_{1,r}A_{2,s} = A_{2,s}a_{1,r}, & 1 \leq r,s \leq 2g;\ r \neq s; \\
\text{(R5)} & (a_{1,1}\cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1}\cdots a_{1,r}), & 1 \leq r \leq 2g; \\
\text{(R6)} & a_{1,r}\sigma_i = \sigma_i a_{1,r}, & 1 \leq r \leq 2g;\ i \geq 2.
\end{array}
$$

Where:

$$
\begin{aligned}
t_{1,j} &= \sigma_1\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}, \text{ for } j = 2,\ldots,n, \\
A_{2,s} &= \sigma_1^{-1}(a_{1,1}\cdots a_{1,s-1}a_{1,s+1}^{-1}\cdots a_{1,2g}^{-1})\sigma_1^{-1}, \text{ for } s = 1,\ldots,2g.
\end{aligned}
$$

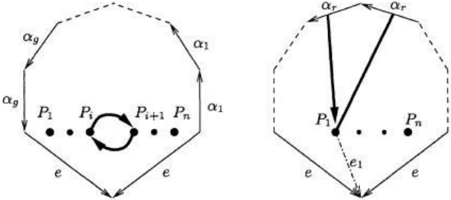
Remark

In the sequel follows the calculations for M closed, connected and non–orientable surface of genus $g \geq 2$.



Generators of $B_n(M)$

From the left to the right the generators σ_i and $a_{1,r}$.



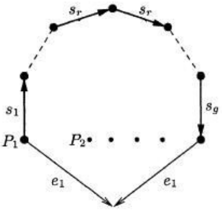
Summary

Braid Groups on Surfaces

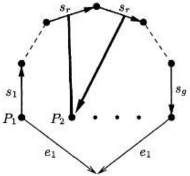
Relations in $B_n(M)$

Presentation of $B_n(M)$

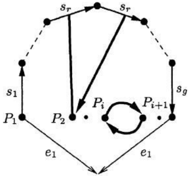
- $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \leq i \leq n-2$,
  holds since $B_n \subseteq B_n(M)$ when $M \neq \mathbb{RP}^2$.
- $a_{1,1}^2 \dots a_{1,g}^2 = \sigma_1 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\dots\sigma_1$ holds since:



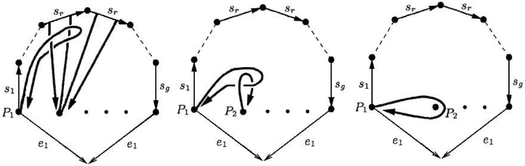
- $a_{1,s}A_{2,r} = A_{2,r}a_{1,s}$, $1 \leq r, s \leq 2g$; $r \neq s$, since:



- $a_{1,r}\sigma_i = \sigma_i a_{1,r}$, $1 \leq r \leq 2g$; $i \geq 2$, since:



- $(a_{1,1}^2 \dots a_{1,r-1}^2 a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1}^2 \dots a_{1,r-1}^2 a_{1,r}), 1 \leq r \leq g$ holds since:



Summary

Theorem (Gonzalez-Meneses [GM, Theorem 2.2, p.436])

Let M be a closed, connected and orientable surface of genus $g \geq 1$. Then, $B_n(M)$ admits the following presentation:
Generators: $\sigma_1, \dots, \sigma_{n-1}, a_{1,1}, \dots, a_{1,2g}$
Relations:

(R1) $\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2$;

(R2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq n-2$;

(R3) $a_{1,1}^2 \dots a_{1,g}^2 = \sigma_1 \dots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \dots \sigma_1$;

(R4) $a_r A_{2,s} = A_{2,s} a_r, 1 \leq r, s \leq g;\ r \neq s$;

(R5) $(a_{1,1}^2 \dots a_{1,r-1}^2 a_{1,r}) A_{2,r} = \sigma_1^2 A_{2,r} (a_{1,1}^2 \dots a_{1,r-1}^2 a_{1,r}), 1 \leq r \leq g$;

(R6) $a_{1,r} \sigma_i = \sigma_i a_{1,r}, 1 \leq r \leq g;\ i \geq 2$.

where $A_{2,r} = \sigma_1^{-1}(a_{1,1}^2 \dots a_{1,r-1}^2 a_{1,r}^{-1} a_{1,r-1}^{-2} \dots a_{1,1}^{-2}) \sigma_1$.

- The figures about braid groups on the disk can be found in [LH]. The others can be found in [GM].

Reference

E. Artin, *Theory of braids*, Ann. of Math., 48 (1946), 101 – 126.

J. González–Meneses, *New Presentation of Surface Braid Groups*, J. of Knot Theory and Its Ramifications, Vol. 10, *n*°. 3, (2001), 431 – 451.

V. L. Hansen, *Braids and Coverings: Selected Topics*, Cambridge University Press, 1989.

Thank You!