Introduction to Braid Groups on Surfaces

Juliana R. Theodoro de Lima
University of São Paulo, Brazil

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Summary

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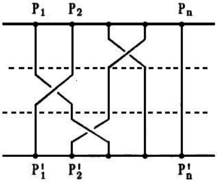
Artin Braid Groups

A *geometric braid in* n *strings* β is a system of n embedded arcs $A = \{A_1, \ldots, A_n\}$ in $\mathbb{E}^3$, where the i-th arc A_i connects the point P_i on the upper plane to the point $P'_{\tau(i)}$ on the lower plane, for some permutation τ de $\{1, \ldots, n\}$, satisfying:

- Each arc A_i intersects each intermediate parallel plane between the upper and the lower plane exactly once;
- The arcs $\{A_1, \ldots, A_n\}$ intersect each intermediate parallel plane between the upper and the lower plane in exactly n different points.

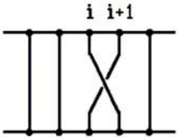


Artin's Presentation Theorem ([A])

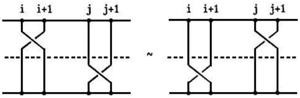
The braid group B_n admits the following presentation:

- **Generators** : $\sigma_1, \ldots, \sigma_{n-1}$.
- **Relations** :

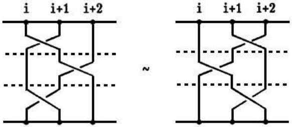
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2, \ 1 \leq i,j \leq n-1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2.$$



Artin Braid Groups



$\sigma_i \cdot \sigma_j$ $\qquad$ $\sigma_j \cdot \sigma_i$



$\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i$ $\qquad$ $\sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$

Summary

Braid Groups on Surfaces – Definition

Let M be a closed connected surface, not necessarily orientable, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of M. A geometric braid over M based at $\mathcal{P}$ is an n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of paths, $\gamma_i : [0,1] \to M \times [0,1]$, such that:

(1) $\gamma_i(0) = P_i,$ for all $i = 1, \ldots, n$,

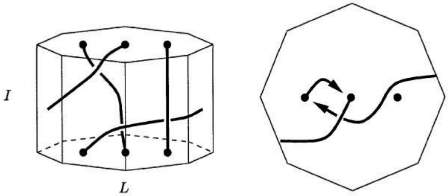
(2) $\gamma_i(1) \in \mathcal{P}$, for all $i = 1, \ldots, n$,

(3) $\{\gamma_1(t), \ldots, \gamma_n(t)\},$ are n distinct points of M, for all $t \in [0,1]$.

For all $i = 1, \ldots, n$, we will call γ_i the i-th string of γ.

Braid Diagrams

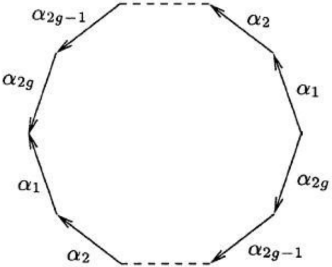
Example of a braid on 3-strands in two different views:



- Two geometric braids based at $\mathcal{P}$ are equivalent if there is a homotopy which deforms one of them into the other, provided that at anytime we always have a geometric braid based at $\mathcal{P}$.
- The product of two braids is induced by the usual product of paths and it endows the set of equivalence classes of braids with a group structure $B_n(M)$.
- If $\gamma_i(1) = P_i,$ for all $i = 1, \dots, n$ then we say that γ is a pure braid. It endows the group $PB_n(M)$ which is a normal subgroup of $B_n(M)$.

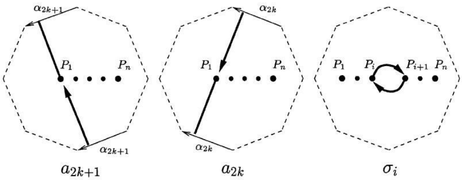
Remark

In the sequel follows the calculations for M closed, connected and orientable surface of genus $g \geq 1$.



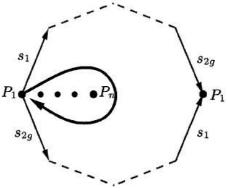
Generators of $B_n(M)$

From the left to the right: $a_{1,2k+1}$, $a_{1,2k}$ and σ_i.

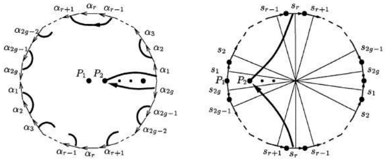


Summary

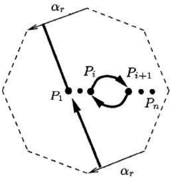
- $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \leq i \leq n-2$,
  holds since $B_n \subseteq B_n(M)$ when $M \neq \mathbb{S}^2$.
- $a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1} = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1$ holds since:



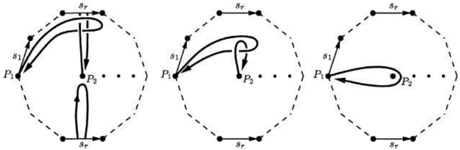
- $a_{1,r}A_{2,s} = A_{2,s}a_{1,r}$, $1 \leq r, s \leq 2g$; $r \neq s$, since:



- $a_{1,r}\sigma_i = \sigma_i a_{1,r}$, $1 \leq r \leq 2g$; $i \geq 2$, since:



- $(a_{1,1} \cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1} \cdots a_{1,r})$, $1 \leq r \leq 2g$ holds since:



Summary

Theorem (Gonzalez-Meneses [GM, Theorem 2.1, p.435])

Let M be a closed, connected and orientable surface of genus $g \geq 1$. Then, $B_n(M)$ admits the following presentation:
Generators: $\sigma_1, \ldots, \sigma_{n-1}, a_{1,1}, \ldots, a_{1,2g}$
Relations:

(R1) $\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2.$

(R2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leq i \leq n-2.$

(R3) $a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1} = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1.$

(R4) $a_{1,r}A_{2,s} = A_{2,s}a_{1,r}, \qquad 1 \leq r,s \leq 2g;\ r \neq s.$

(R5) $(a_{1,1}\cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1}\cdots a_{1,r}), \qquad 1 \leq r \leq 2g.$

(R6) $a_{1,r}\sigma_i = \sigma_i a_{1,r}, \qquad 1 \leq r \leq 2g;\ i \geq 2,$

where $A_{2,r} = \sigma_1^{-1}(a_{1,1}\cdots a_{1,r}a_{1,r+1}^{-1}\cdots a_{1,2g}^{-1})\sigma_1^{-1}$.

- The figures about braid groups on the disk can be found in [LH]. The others can be found in [GM].
- In the same paper, the author found the presentation for M a closed, connect and non–orientable surface of genus $g \geq 2$.

Reference

- E. Artin, *Theory of braids*, Ann. of Math., 48 (1946), 101 – 126.
- J. González–Meneses, *New Presentation of Surface Braid Groups*, J. of Knot Theory and Its Ramifications, Vol. 10, n^o. 3, (2001), 431 – 451.
- V. L. Hansen, *Braids and Coverings: Selected Topics*, Cambridge University Press, 1989.

Thank You!