The Hyperbolic Metric in Complex Analysis

Eric Schippers

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Geometric function theory

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Examples:

- Relation between the shape of the image domain and the analytic properties of a function.
- Analytic properties which guarantee the functions are one-to-one.
- Moduli spaces of Riemann surfaces.
- Distribution of zeroes (value distribution theory).
- Hyperbolic geometry of analytic functions - special case of study of "conformal metrics".

Definition of conformal metric

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Definition

The **ρ-length** of a curve γ in D is

$$\mathcal{L}_\rho(\gamma) = \int_\gamma \rho(z)|dz|.$$

$$|dz| = \left|\frac{dz}{dt}\right| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

for $z(t) = x(t) + iy(t)$

Terminology

- A conformal metric is a special type of “Riemannian metric”.
- The term “metric” here is not the same as the term in analysis.
- However, every conformal metric gives rise to a distance function, which is in fact a metric.

Hyperbolic metric

$\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$

Definition

The **hyperbolic metric** on $\mathbb{D}$ is

$$\lambda(z) = \frac{1}{1 - |z|^2}.$$

The **hyperbolic length** of a curve γ in $\mathbb{D}$ is

$$\mathcal{L}(\gamma) = \int_\gamma \frac{|dz|}{1 - |z|^2}.$$

This is one example of a hyperbolic metric.

Isometries

Definition

An **isometry** is a one-to-one onto map $f : \mathbb{D} \to \mathbb{D}$ such that for any curve γ,

$$\mathcal{L}(f \circ \gamma) = \mathcal{L}(\gamma).$$

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Möbius transformations $T : \mathbb{D} \twoheadrightarrow \mathbb{D}$ are isometries of the hyperbolic metric:

$$T(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} \quad \Rightarrow \quad \frac{|T'(z)|}{1 - |T(z)|^2} = \frac{1}{1 - |z|^2}.$$

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so

$$\mathcal{L}(T \circ \gamma) = \int_{T\circ\gamma} \frac{|dw|}{1-|w|^2} = \int_\gamma \frac{|T'(z)||dz|}{1-|T(z)|^2} = \int_\gamma \frac{|dz|}{1-|z|^2} = \mathcal{L}(\gamma).$$

Isometries continued

In fact these are all of them!

Wonderful coincidence:

$$\begin{aligned}\{\text{Isometries}\} &= \left\{ T(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z} \; : \; \theta \in \mathbb{R}, \; a \in \mathbb{D} \right\} \\ &= \{\text{one-to-one}, \; \text{onto}, \; \text{analytic } T : \mathbb{D} \to \mathbb{D}\}.\end{aligned}$$

hyperbolic distance

Definition

The **hyperbolic distance** between two points z and w is

$$d(z,w) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1-|z|^2}.$$

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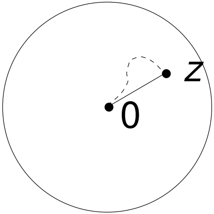
Definition

A **geodesic segment** between two points is a curve which attains the minimum distance.

Warning: This is not the usual definition, but in the case of the hyperbolic metric on the disc, it is equivalent to the usual one.

Geodesics and the distance function

Shortest path from 0 to z is the radial line segment:



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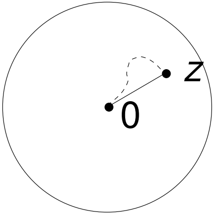


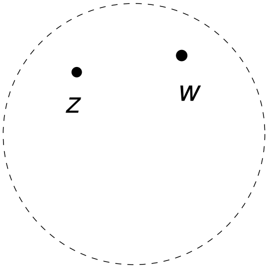
So the hyperbolic distance between 0 and z is:

$$\begin{aligned} \int_{\text{line}} \frac{|dz|}{1-|z|^2} &= \int_0^{|z|} \frac{dr}{1-r^2} \\ &= \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right) = \operatorname{arctanh}|z|. \end{aligned}$$

Geodesics and the distance function

So we can determine the shortest path through any points z and w: let

$$T(\zeta) = \frac{\zeta - w}{1 - \bar{\zeta} w}.$$



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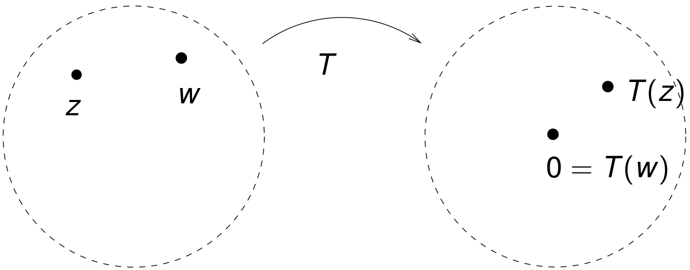

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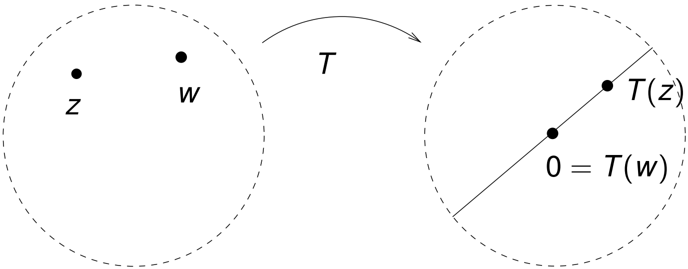

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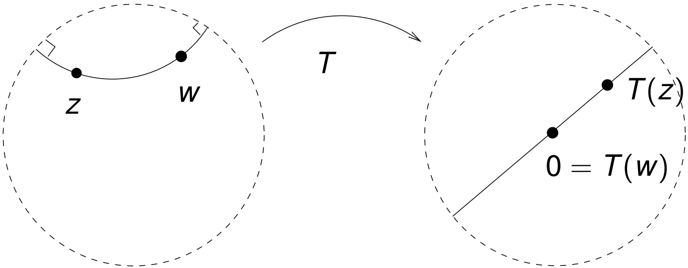
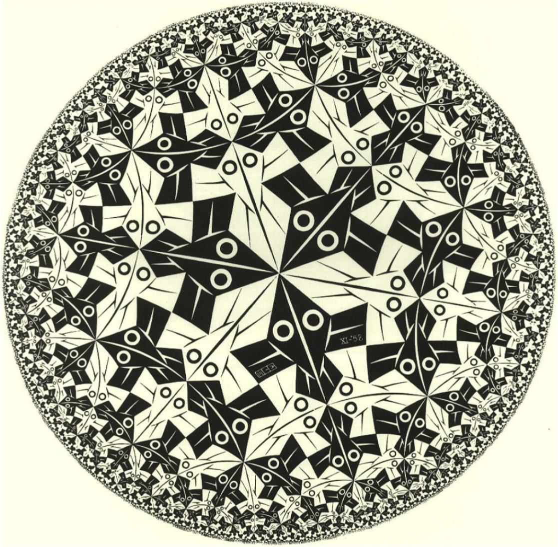


Geodesics and the distance function

So we can in turn determine the distance between z and w:

$$\begin{aligned} d(z,w) &= d(T(z),T(w)) = d(T(z),0) = \operatorname{arctanh}\left|\frac{z-w}{1-\bar{w}z}\right| \\ &= \frac{1}{2}\log\frac{1+\left|\frac{z-w}{1-\bar{w}z}\right|}{1-\left|\frac{z-w}{1-\bar{w}z}\right|}. \end{aligned}$$

Hyperbolic world



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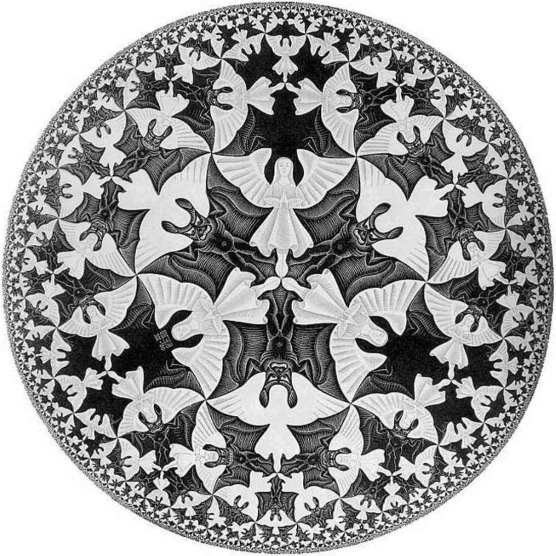

Curvature

Definition

Let D be a domain in $\mathbb{C}$. Let $\rho(z) : D \to \mathbb{R}^+$ be a conformal metric. The curvature of ρ is

$$K(z) = -\frac{1}{\rho^2(z)} \Delta \log \rho(z).$$

This is a special case of a more general notion in differential geometry.

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What is curvature?

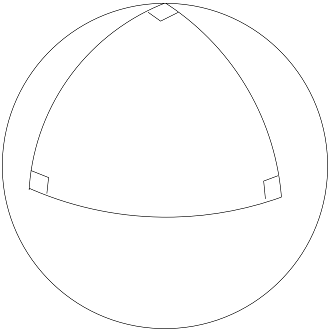
Theorem (Gauss-Bonnet theorem (special case))

The sum of the interior angles of a triangle D is

$$\pi + \iint_D K dA_\rho.$$

Example: geodesic triangles on the sphere

$\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$: geodesics are great circles, $K = 1$



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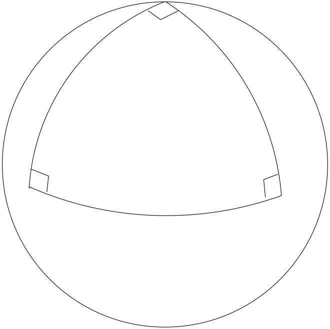

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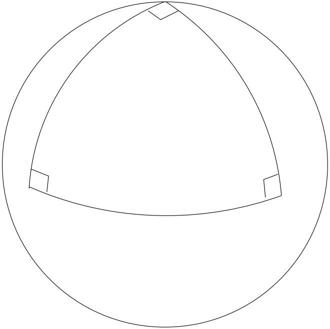

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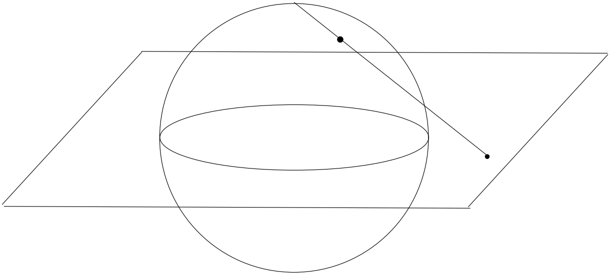
$\pi + \iint K dA_\rho = \pi + 1 \cdot \text{Area} = \pi + \pi/2$.

A bit more detail

I didn't define curvature in enough generality to justify that last bit. You can use the definition I gave if you stereographically project:



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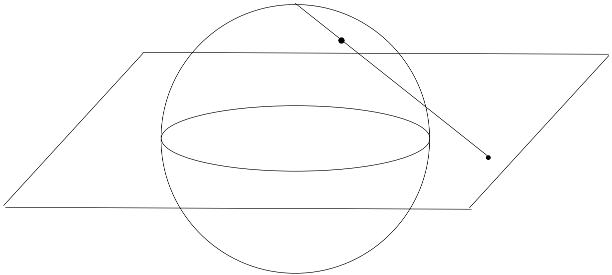

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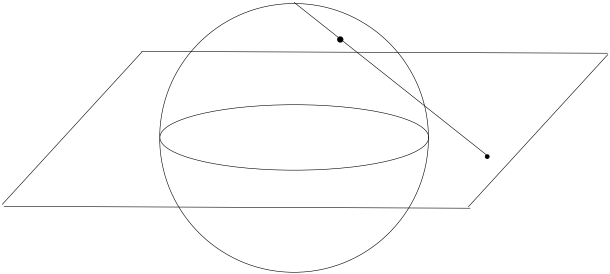


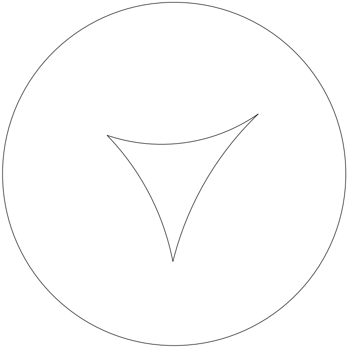
(1) Trigonometry + work shows: the length of a curve on the sphere, is the ρ-length of the projected curve if $\rho(z) = 2/(1+|z|^2)$
(2) the ρ-area on the plane is the usual area on sphere
(3) the curvature of ρ is 1.

Hyperbolic case

If $\lambda(z) = 1/(1-|z|^2)$, then curvature is -4:

$$\begin{aligned} K(z) &= -\frac{4}{\rho^2(z)} \frac{\partial^2}{\partial z \partial \bar{z}} \log \rho \\ &= 4(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} \log (1 - z\bar{z}) \\ &= -4. \end{aligned}$$

Hyperbolic triangles



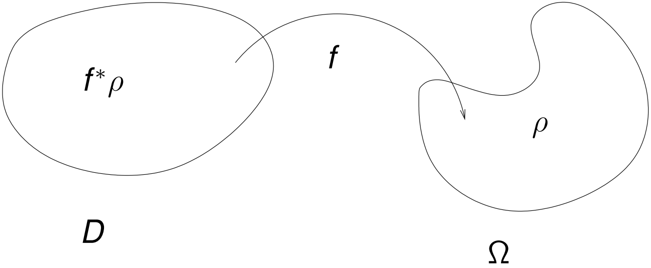
Sum of angles $< \pi$.

Pull-back

Definition

Let ρ be a metric on a domain $\Omega \subset \mathbb{C}$. Let $f : D \to \Omega$ be an analytic map such that $f' \neq 0$. The pull-back of ρ under f is

$$f^*\rho(z) = \rho \circ f(z)\,|f'(z)|.$$



Example

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The pull-back of the hyperbolic metric $\lambda(z) = 1/(1 - |z|^2)$ on $\mathbb{D}$ to $\mathbb{D}_R$

is

$$\begin{aligned} f^*\lambda(z) &= \lambda(f(z))|f'(z)| = \frac{1/R}{(1 - |z/R|^2)} \\ &= \frac{R}{R^2 - |z|^2}. \end{aligned}$$

Idea of pull-back

Idea: the pull-back geometry on D is “the same” as the geometry on Ω.

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Length is preserved: if γ is a curve in D

$$\rho\text{-length}(f \circ \gamma) = \int_{f\circ\gamma} \rho(z)|dz| = \int_{\gamma} \rho(f(z))|f'(z)||dz| = f^{*}\rho\text{-length}(\gamma).$$

Curvature is preserved:

$$K_{f^*\rho}(z) = K_{\rho}(f(z)).$$

Try it!

Completeness

Definition

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Theorem

The hyperbolic metric on $\mathbb{D}$ is complete.

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Let D be a domain in the plane. The hyperbolic metric of D is the unique complete metric on D with constant negative curvature -4 (provided that it exists).

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Example: The hyperbolic metric on $\mathbb{D}$ is $\lambda(z) = 1/(1-|z|^2)$.

Example: The hyperbolic metric on $\mathbb{D}_R$ is

$$\lambda_R(z) = \frac{R}{R^2 - |z|^2}.$$

Why? Because λ_R is the pull-back of the hyperbolic metric, and so it is complete and constant curvature -4.

Uniformization theorem

Theorem (Uniformization theorem (Koebe, Poincaré))

Every simply connected Riemann surface is biholomorphically equivalent to the Riemann sphere $\overline{\mathbb{C}}$*, the complex plane* $\mathbb{C}$*, or the unit disk* $\mathbb{D}$*.*

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Corollary

Every Riemann surface is given by the quotient of $\overline{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$ by a nice group action (think tiling).

Almost everything is covered by $\mathbb{D}$

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- $\mathbb{C}$ only covers $\mathbb{C}$, $\mathbb{C}\backslash\{0\}$ (cylinder) and tori.
- **Everything else** is $\mathbb{D}/G$ for some nice subgroup of the Möbius transformations of the form

$$T(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \quad a \in \mathbb{D}...$$

which are hyperbolic isometries. So almost every Riemann surface has a hyperbolic metric inherited from Δ.

Uniformization theorem

Nearly all domains have a hyperbolic metric.

Corollary (Uniformization theorem)

Any subset of the plane which omits at least two points possesses a hyperbolic metric.

Where's the complex analysis?

The isometries of the hyperbolic metric are exactly the conformal automorphisms of the disc.

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So there should be some connection between complex analysis on the disc and hyperbolic geometry on the disc.
Let's look at some examples.

Schwarz lemma

Theorem

If $f : \mathbb{D} \to \mathbb{D}$ *is analytic and* $f(0) = 0$ *then* $|f(z)| \leq |z|$.

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Analytic maps from $\mathbb{D}$ to $\mathbb{D}$ are contractions.

proof of hyperbolic Schwarz lemma

Let

$$T(z) = \frac{z + w}{1 + \bar{w}z}, \quad S(\zeta) = \frac{\zeta - f(w)}{1 - \overline{f(w)}\zeta}.$$

So if $f : \mathbb{D} \to \mathbb{D}$ then $S \circ f \circ T(0) = S(f(w)) = 0$.

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So if $f : \mathbb{D} \to \mathbb{D}$ then $S \circ f \circ T(0) = S(f(w)) = 0$.
By the Schwarz lemma, $|S(f(T(z)))| \leq |z| \Rightarrow |S(f(z))| \leq |T^{-1}(z)|$ so

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} \right| \leq \left| \frac{z-w}{1-\bar{w}z} \right|.$$

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But arctanh is increasing so

$$d(f(z), f(w)) = \operatorname{arctanh} \left| \frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} \right| \leq \operatorname{arctanh} \left| \frac{z-w}{1-\bar{w}z} \right| = d(z,w).$$

Ahlfors' generalization of the Schwarz lemma

Theorem (Ahlfors-Schwarz lemma, special case)

Let $\mathbb{D}_R$ be the disc of radius R, with hyperbolic metric λ_R. For any metric ρ on $\mathbb{D}_R$, such that the curvature $K_\rho(z) \leq -4$ for all z,

$$\rho(z) \leq \lambda_R(z)$$

for all z.

The proof

Proof.

For $r < R$ we have $\mathbb{D}_r \subset \mathbb{D}_R$. Let

$$v(z) = \frac{\rho}{\lambda_r}; \quad z \in \mathbb{D}_r.$$

v is continuous, positive, and $v \to 0$ as $|z| \to r$.

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$$\begin{aligned} 0 &\geq \triangle \log v(z_0) = \triangle \log \rho - \triangle \log \lambda_r \\ &= -\rho^2(z_0) K_\rho(z_0) + \lambda_r^2(z_0) K_{\lambda_r}(z_0) \\ &\geq 4\rho^2(z_0) - 4\lambda_r^2(z_0). \end{aligned}$$

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So since z_0 was the maximum, $\rho(z) \leq \lambda_r(z)$ for all z.
Now let $r \to R$. □

Ahlfors

Ahlfors 1907-1996

According to Ahlfors: “This is an almost trivial fact and anyone who sees the need could prove it at once”.

Ahlfors continued

- Finnish mathematician, advisors at University of Helsinki were E. Lindelöf and R. Nevanlinna
- First Fields Medal (with Jesse Douglas) in 1936 for work in value distribution theory (Nevanlinna theory).
- Wolf Prize in 1981
- Towering figure in Riemann surfaces and complex analysis
- Most famous as one of the founders of Teichmüller theory and quasiconformal mappings

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for all z.

Ahlfors' generalization of the Schwarz lemma

Theorem (Ahlfors-Schwarz lemma, special case)

Let $\mathbb{D}_R$ be the disc of radius R, with hyperbolic metric λ_R. For any metric ρ on $\mathbb{D}_R$, such that the curvature $K_\rho(z) \leq -4$ for all z,

$$\rho(z) \leq \lambda_R(z)$$

for all z.

It says:

- The hyperbolic metric is maximal, among metrics with bounded negative curvature
- In particular, if $f : \mathbb{D}_R \to \Omega$, and λ_R is the hyperbolic metric on $\mathbb{D}_R$, and λ_Ω is the hyperbolic metric on Ω, then $f^*\lambda_\Omega \leq \lambda_R$.

Schwarz lemma is special case

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Let $f : \mathbb{D} \to \mathbb{D}$ and $\lambda(z) = 1/(1 - |z|^2)$ be the hyperbolic metric.

The curvature of $f^*\lambda$ equals -4 since curvature is invariant under pull-back.

So by the Ahlfors-Schwarz lemma

$$f^*\lambda \leq \lambda$$

so

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Liouville's theorem

Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ *be an analytic function such that* $|f(z)| \leq M$ *for all* $z \in \mathbb{C}$*. Then* f *is constant.*

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Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is constant.

Liouville's theorem can be interpreted as a limiting case of Schwarz lemma.

Proof of Liouville's theorem using the Schwarz lemma

Minda, Schober 1983.

The hyperbolic metric on the disc of radius R is

$$\lambda_R(z) = \frac{R}{R^2 - |z|^2}.$$

For any R, f maps $\mathbb{D}_R$ into $\mathbb{D}_M$.

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For any R, f maps $\mathbb{D}_R$ into $\mathbb{D}_M$.
By the Ahlfors-Schwarz lemma, for any R,

$$f^* \lambda_M \leq \lambda_R$$

so

$$\frac{M|f'(z)|}{M^2 - |f(z)|^2} \leq \frac{R}{R^2 - |z|^2}$$

for any fixed z.

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so

$$\frac{M|f'(z)|}{M^2 - |f(z)|^2} \leq \frac{R}{R^2 - |z|^2}$$

for any fixed z.
Letting $R \to \infty$, we get that $|f'(z)| = 0$ for any z. So $f = c$. □

The Little Picard Theorem

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Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic mapping, whose image omits at least two points. Then f is constant.

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Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic mapping, whose image omits at least two points. Then f is constant.

The Little Picard theorem is really a case of the (Ahlfors-)Schwarz lemma in disguise.

proof of the Little Picard Theorem

Proof.

Let p, q be the points omitted from the image of f. Let σ be the hyperbolic metric on $\mathbb{C}\backslash\{p, q\}$ (uniformization theorem!)

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$$\sigma(f(z))|f'(z)| = f^*\sigma \leq \lambda_R(z) = \frac{R}{R^2 - |z|^2}.$$

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This approach due to Minda and Schober (1983). Actually this is a variation on the classical approach. They also give an elementary proof without using the uniformization theorem.

Summary of hyperbolic complex analysis theorems

- The Schwarz lemma *really* says that analytic maps from $\mathbb{D}$ to $\mathbb{D}$ are hyperbolic contractions.
- Liouville's theorem is really a limiting case of the Schwarz lemma.
- The Little Picard Theorem is really a limiting case of the Schwarz lemma.
- Actually, any holomorphic map between hyperbolic Riemann surfaces is a hyperbolic contraction.
- The hyperbolic metric is central to complex analysis.

Some References

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