Introduction to group growth.

We often speak about finitely generated groups by giving a presentation

$$\Gamma = \langle s_1, \dots, s_n \mid r_1, r_2, \dots \rangle$$

This notation means we're taking the free group $F(S)$ on $S = \{s_1, \dots, s_n\}$, and Γ is the quotient of $F(S)$ by the smallest normal subgroup containing $r_1, r_2, \dots$ etc.

So it's very common to specify Γ by giving $F(S) \to \Gamma$.

Definition: Let Γ be the quotient of $F(S)$ under the map $q : F(S) \longrightarrow \Gamma$, $(|S| < \infty)$

The word length $\ell_S(\gamma)$ of $\gamma \in \Gamma$ is the smallest k for which $\exists$ a product

$$s_{i_1} s_{i_2} \cdots s_{i_k} \in F(S), \quad s_{i_j} \in S \cup S^{-1}$$

such that $\gamma = q(s_{i_1} s_{i_2} \cdots s_{i_k})$, with $\gamma_S(id) = 0$.

We can make Γ into a metric space by setting

$$d_S(\gamma_1, \gamma_2) = \ell_S(\gamma_1^{-1}\gamma_2),$$

then this distance is simply distance in the Cayley graph $Cay(\Gamma, S)$:

vertices $= \Gamma$, with γ_1, γ_2 connected by an edge iff $d(\gamma_1, \gamma_2) = 1$.

Then set

$$\beta(\Gamma, S; k) = |\{\gamma \in \Gamma \mid l_S(\gamma) \le k\}| = \left|\begin{matrix}\text{Ball of radius} \\ k \text{ in } Cay(\Gamma, S)\end{matrix}\right| \in \mathbb{Z}.$$

So we're counting elements of length $\le k$; in order to gather information about the structure of Γ we'll analyze the $\beta(\Gamma, S; k)$'s.

<u>Example</u>: Consider $\Gamma = (\mathbb{Z}, +)$ and $S = \{1\}$.

Then $\beta(\mathbb{Z}, \{1\}; 1) = 3$, since $-1, 0, 1$ are the only elements of $\mathbb{Z}$ that are a sum of ≤ 1 1's.

In general, $\beta(\mathbb{Z}, \{1\}; k) = 1 + 2k$

On the other hand, consider $\mathbb{Z}$ with $S = \{2, 3\}$. Then

$\beta(\mathbb{Z}, \{2,3\}; 1) = 5$, since we can get $\pm 2, \pm 3, 0$ taking sums of ≤ 1 elts from $\{2, 3\}$.

$\beta(\mathbb{Z}, \{2,3\}; 2) = 5 + 8$, since we get 8 new elements upon doing

$$\begin{matrix} \pm 2 \pm 2 \\ \pm 2 \pm 3 \\ \pm 3 \pm 3 \\ \pm 3 \pm 2. \end{matrix}$$

In general, $\beta(\mathbb{Z}, \{2,3\}; k) = 5 + 8 + 6(k-2)$, because going from k to $k+1$ when $k \geq 2$ adds 6 new elements:

$\pm(3k-2), \pm(3k-1), \pm 3k$ (can prove this by induction)

Moral of example: The values $\beta(\Gamma, S; k)$ depend on our choice of S, even $\Gamma = \mathbb{Z}$ gives problems if we want to use the $\beta(\Gamma, S; k)$'s to learn something about Γ.

So we create something new:

Def: The growth series of a group Γ is the power series

$$B(\Gamma, S; z) = \sum_{k=0}^{\infty} \beta(\Gamma; S; k) z^k.$$

People often also set

$$\sigma(\Gamma, S; k) = \beta(\Gamma, S; k) - \beta(\Gamma, S; k-1)$$

and define the spherical growth series to be

$$\Sigma(\Gamma, S; z) = \sum_{k=0}^{\infty} \sigma(\Gamma; S; k) z^k$$
$$= (1-z) B(\Gamma, S; z)$$

(This is since the number of elements added when passing from length $k-1$ to length k is sometimes more natural to work with).

Example: If $\Gamma = \mathbb{Z}$ and $S = \{1\}$, then

$$\Sigma(\mathbb{Z}, S; z) = 1 + \sum_{k=1}^{\infty} 2z^k = \frac{1+z}{1-z}, \text{ so}$$

$$B(\mathbb{Z}, S; z) = \frac{1+z}{(1-z)^2}.$$

On the other hand, if $S = \{2, 3\}$ then

$$\Sigma(\mathbb{Z}, S; z) = 1 + 4z + 8z^2 + 6z^3 + 6z^4 + \dots = \frac{1+3z+4z^2-2z^3}{1-z}.$$

Examples can be difficult to compute, though there are a few theorems to help. Eg.

<u>Theorem</u>: If S_1 generates Γ_1, S_2 generated Γ_2, then

$$\Sigma(\Gamma_1 \times \Gamma_2, S_1 \times \{1\} \cup \{1\} \times S_2; z) = \Sigma(\Gamma_1, S_1; z) \cdot \Sigma(\Gamma_2, S_2; z)$$

So, e.g $\quad \Sigma(\mathbb{Z}^n; \text{standard basis}; z) = \left(\frac{1+z}{1-z}\right)^n$.

- We still have the problem that different generating sets of a group give different (spherical or not) growth functions.
- We need to correct the problem, this is where the "geometry" of geometric group theory comes in to play.

<u>Question</u>: Suppose that S, S' are both finite generating sets of Γ. How are $Cay(\Gamma, S)$ and $Cay(\Gamma, S')$ related? (I.e, how are l_S and $l_{S'}$ related?).

Definition: Let (X,d) and (X',d') be (pseudo) metric spaces. A map $\phi: X \longrightarrow X'$ is called a quasi-isometric embedding if $\exists \lambda \geq 1, C \geq 0$ such that

$$\frac{1}{\lambda} d(x,y) - C \leq d(\phi(x), \phi(y)) \leq \lambda d(x,y) + C.$$

$\forall x, y \in X$. If $\exists D \geq 0$ s.t. $\forall x' \in X' \exists x \in X$ with $d(x,x') \leq D$, then X and X' are quasi-isometric.

(Equivalently, $\exists$ QI embeddings $\phi: X \to X'$ and $\phi': X' \longrightarrow X$).

Examples:

① Every metric space of finite radius is QI to a point.

② (Theorem):

$Cay(\Gamma, S)$ and $Cay(\Gamma, S')$ are quasi-isometric.

Proof: Given two presentations of Γ:

$$\Gamma = \langle S \mid R \rangle \quad \text{and} \quad \Gamma = \langle S' \mid R' \rangle,$$

write an isomorphism $\phi: \Gamma \longrightarrow \Gamma$ taking one to the other. Show this induces a QI

$$Cay(\Gamma, S) \longrightarrow Cay(\Gamma, S').$$

Note: Different groups can be quasi-isometric, for instance all finite groups are quasi-isometric to $\{1\}$.

What equivalence relation on growth functions does this give?

Def: A growth function B_2 weakly dominates B_1 if $\exists\ \lambda \geq 1,\ C \geq 0$ s.t.

$$B_1(z) \leq \lambda B_2(\lambda z + C) - C$$

for all $z \in \mathbb{R}^+$. We write $B_1 \prec B_2$.

If $B_1 \prec B_2$ and $B_2 \prec B_1$, then $B_1 \sim B_2$. (They are weakly equivalent).

Theorem: Let (Γ_1, S_1) and (Γ_2, S_2) be infinite groups, $|S_1|, |S_2| < \infty$. If $Cay(\Gamma_1, S_1)$ quasi-isometrically embeds in $Cay(\Gamma_2, S_2)$, then

$$B_1(\Gamma_1, S_1; z) \prec B_2(\Gamma_2, S_2; z).$$

In particular if S, S' are different generating sets of Γ, then

$$B(\Gamma, S, z) \sim B(\Gamma, S', z).$$

So we have

$$\text{Groups}/\text{Quasi-isometry} \longmapsto \text{Growth functions}/\text{weak equivalence}.$$

and in particular, many groups map to the same thing.

E.g.

Nilpotent groups $\longmapsto$ polynomials
$\left(t^a \prec t^b \text{ iff } a<b, \text{ and } \alpha(t) \text{ degree } d \Rightarrow \alpha(t) \sim t^a \text{ iff } d=a \right)$

Finitely generated free groups (all QI) $\longmapsto$ exponentials
(all equivalent, ie. $e^{at} \sim e^{bt}$ $\forall a,b>0$)

Question (Milnor 1968):

Can anyone find a group that isn't polynomial or exponential growth?

(Grigorchuk 1984)

Theorem: There are uncountably many finitely generated groups, all of whose growth functions are pairwise non-comparable (neither $B_1 \prec B_2$ or $B_2 \prec B_1$ holds).

In particular, some examples must have neither polynomial nor exponential growth.