Serhii

Dehn Surgery (Introduction).

<u>Def</u>: A knot is a piecewise linear simple closed curve in S^3.

During this talk, if $K \subset S^3$, then $V(K)$ will denote a tubular nbhd of K (So $V(K) \simeq D^2 \times S^1$).

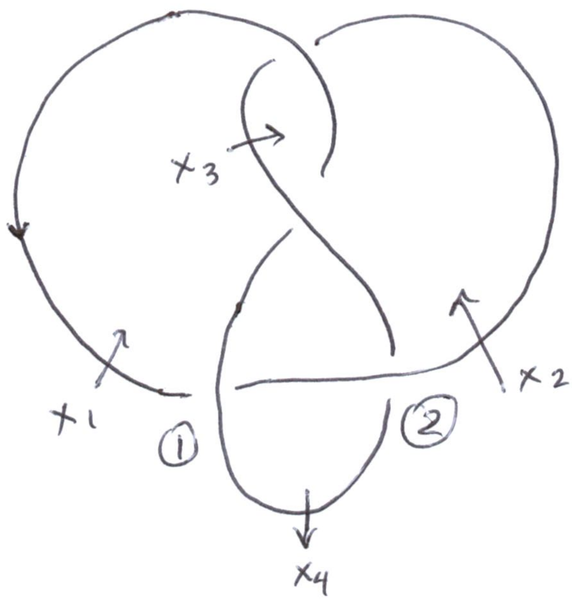
<u>Def</u>: A <u>knot complement</u>, or the <u>complement</u> of a knot $K \subset S^3$, is $K \setminus V(K)$.

<u>Remark</u>: In S^3, a knot K determines a 3-manifold, which is compact, by taking $S^3 \setminus V(K)$.

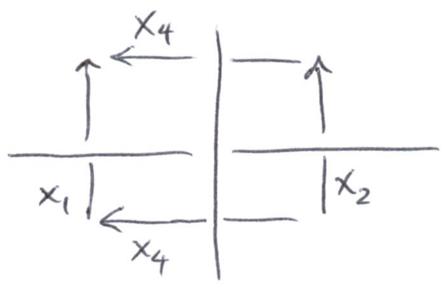
<u>Def</u>: The <u>knot group</u> of a knot K is the fundamental group of $S^3 \setminus V(K)$.

Assume that every knot admits a diagram with only double points, called crossings. Omitting this technical detail, here is an example of how to compute $\pi_1(S^3 \setminus V(K))$ using the Wirtinger presentation.

First, start with a diagram:

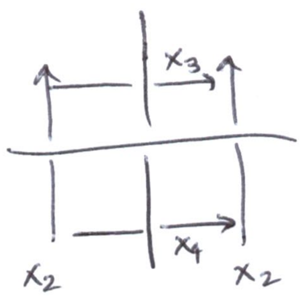


Crossing 1 gives:



$\Rightarrow x_2 = x_4 x_1 x_4^{-1}.$

2



$\Rightarrow x_4 = x_2 x_3 x_2^{-1}$

3 Gives $x_2 = x_1 x_3 x_1^{-1}$.

4 We do not need to compute the relation corresponding to the 4th crossing, there is a theorem which says that any one crossing is a consequence of the others.

So

$$\pi_1(S^3 \setminus (4_1)) = \langle x_1, x_2, x_3, x_4 \mid \text{relations } ①, ②, ③ \rangle.$$

<u>Dehn surgery</u> is when you remove a tubular nbhd of $K \subset S^3$, and replace it with a torus which is glued in differently.

Then $V(K) \cong S^1 \times D^2$

$S^1 \times \{0\}$ corresponds to K,

and $\lambda \sim S^1 \times \{1\}$ is called a longitude of K.

The defining property of λ is that we want λ to be trivial in $H_1(S^3 \setminus K)$.

To see why this is possible, observe that

$$\pi_1(S^3 \setminus V(K)) \longrightarrow \mathbb{Z},$$

since upon abelianizing the relations $\cancel{x_i} x_j \cancel{x_i^{-1}} = x_k$

$\Rightarrow x_j = x_k$.

So all relations gives something like $x_j = x_k$.

Then

$$\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \pi_1(S^3 \setminus K) \longrightarrow \mathbb{Z},$$

so there's an element of $\pi_1(T)$ that maps to 0.

That curve is the longitude, we call it the preferred longitude, it is one generator of $\pi_1(T)$.

The other generator of $\pi_1(T)$, called μ, is identified with $\partial D^2 \times \{1\}$.

Then $\langle \mu, \lambda \rangle = \pi_1(T) \subset \pi_1(S^3 \setminus T)$ is the subgroup determined by the boundary torus, and we have chosen a particular basis.

Now fix some curve J which satisfies

- $J \subset \partial(S^1 \times D^2)$
- J is homotopically nontrivial, and is a simple closed curve.

Define the result of Dehn surgery along K to be:

$$M = (S^3 \setminus int(V(K))) \cup_\varphi (S^1 \times D^2)$$

$$\varphi : S^1 \times S^1 \longrightarrow \partial V(K).$$

where $\varphi(\mu) = J$.

Why does such a φ exist?

Well, represent

$$[J] = \mu^p \lambda^q \quad (\text{want } p, q \text{ relatively prime})$$

then observe: the mapping class group of a torus is $MCG(T) = \{\text{homeomorphisms}\}/\text{isotopy} \cong GL(2, \mathbb{Z})$.

Each $h: T \longrightarrow T$

induces $h_*: \pi_1(T) \longrightarrow \pi_1(T)$

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

and this means h_* is represented by a 2×2 matrix.
In fact, $h \longmapsto h_*$ is a bijection (up to isotopy) and we get $MCG(T) \cong GL(2, \mathbb{Z})$.

<u>Theorem</u>: The homeomorphism type of M depends only on $K \subset S^3$ and our fraction $\frac{p}{q} \in \mathbb{Q}$.

The reason for discussing Dehn surgery is the following theorem of Lickorish and Wallace.

By def, our ~~element~~ curve J bounds a disk in M.
$\Rightarrow [J]$ is trivial in the fundamental group of M.

therefore

$$\pi_1(M) = \pi_1(S^3 \setminus \nu(K)) + \text{new relation } \mu^p \lambda^q = 1.$$

(This can also be viewed as an application of the Seifert-Van Rampen theorem).

Theorem: (Lickorish-Wallace)

Every compact, connected 3-manifold is the result of surgery along a link in S^3.