Eric Schippers.

# Introduction to Lie groups

Start with a manifold, i.e. a space locally homeomorphic to $\mathbb{R}^n$ and that has enough structure that the statement

"$f: M \longrightarrow N$ is $C^\infty$ differentiable"

makes sense.

So, for example we require smooth transition maps.

A <u>Lie group</u> is a group $G$ which is also a manifold, such that multiplication and inversion are $C^\infty$ maps.

<u>Ex</u>:
- $\mathbb{R}$ is a Lie group under addition.
- $S^1$ is a Lie group under addition mod $2\pi$ (of the angles).
- $GL(n, \mathbb{R}) = \{ M \in M_{n \times n}(\mathbb{R}) \mid \det(M) \neq 0 \}$, inverses gives a smooth function but it takes some checking.

<u>Note</u>: We drop "$\mathbb{R}$" and assume all matrix groups are over $\mathbb{R}$ from now on

- $O(n) = \{ M \in GL(n, \mathbb{R}) \mid M^T M = I \}$.
- $A(n) = \left\{ \begin{pmatrix} A & \vec{v} \\ 0 & 1 \end{pmatrix} \,\middle|\, A \in O(n), \vec{v} \in \mathbb{R}^n \right\}$, can identify this with a matrix group with action

$$\begin{pmatrix} A & \vec{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + \vec{v} \\ 1 \end{pmatrix},$$ so affine here.

## Motivation from geometry: Klein's Erlangen programme

Geometries are homogeneous spaces and geometric quantities are invariants under some group. So, for example:

Ex: Euclidean geometry is the study of

$\mathcal{A}(n)/O(n) \cong \mathbb{R}^n$. Then given $A \in O(n)$, we get

$\vec{x}^T\vec{x} = (A\vec{x})^T A\vec{x} = \vec{x}^T A^T A \vec{x} = \vec{x}^T\vec{x}$, so length $= \vec{x}^T\vec{x}$

is invariant under the group $O(n)$.

Ex: $O(3,1) = \{ A \in M_{4,4}(\mathbb{R}) \mid A^T \eta A = \eta \}$, $\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

and we study quantities invariant under this group, one of which is $\vec{x}^T \eta \vec{x} = -t^2 + x^2 + y^2 + z^2$.

---

## Exponential maps:

Theorem: For any matrix $A \in M_{n\times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$, define

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

then this converges.

Proof: Set $M = \sup_{j,i} |A_{i,j}|$. Then $|A_{i,j}|^2 \leq nM^2$, and inductively $|A_{i,j}|^k \leq n^{k-1} M^k$.

Then

$$\sum_{k=0}^{\infty} \frac{|A^k_{ij}|}{k!} \leq \sum_{k=0}^{\infty} \frac{n^{k-1}M^k}{k!} = \frac{e^{nM}}{n},$$

so it's a bounded quantity, so the terms converge (absolutely)

<u>Theorem</u>: If $AB = BA$ then $e^B \cdot e^A = e^{A+B}$

<u>Proof</u>: $\left(I + B + \frac{B^2}{2!} + \cdots\right)\left(I + A + \frac{A^2}{2!} + \cdots\right)$

$$= I + (B+A) + \frac{(B+A)^2}{2!} + \cdots$$

Here we're using absolute convergence to reorder terms, and commutativity to regroup terms as we did.

<u>Also observe</u>: $\left.\frac{d}{dt}\right|_{t=0} e^{tB} = B$,

and we have a formula:

$$\det(e^B) = e^{\operatorname{Tr} B}$$

## <u>Lie Algebras</u>:

<u>Definition</u>: A <u>Lie Algebra</u> is a vector space $V$ with a bracket operation

$$[\ ,\ ] : V \times V \longrightarrow V$$

such that

(i) $[v, w] = -[w, v]$

(ii) $[[v,w],u] + [[w,u],v] + [[u,v],w] = 0$
(Jacobi identity).

(iii) The bracket is linear. (in each term).

Idea: Every Lie group has an associated Lie algebra. The Lie algebra is the tangent space of $G$ at the identity. We write a gothic "g": $\mathfrak{g}$.

For this talk,

$$\mathfrak{g} = \left\{ \frac{d}{dt}\Big|_{t=0} A(t) \;\middle|\; \begin{array}{l} A(t) \text{ is a smooth curve of} \\ \text{matrices in a matrix group } G \\ \text{s.t. } A(0) = I \end{array} \right\}.$$

Example: Consider

$$O(n) = \{ A \mid A^T A = 1 \}.$$

Claim: The associated Lie algebra is

$$\mathfrak{o}(n) = \{ A \mid A^T + A = 0 \}.$$

Proof: Let $A(t)$ be a curve such that $A(t)^T A(t) = I$.

Then
$$0 = \frac{d}{dt}\Big|_{t=0} A(t)^T A(t) = \dot{A}(0)^T A(0) + A(0)^T \dot{A}(0)$$
$$= \dot{A}(0)^T + \dot{A}(0) \quad \text{(use dot for derivative).}$$

So the derivatives of curves satisfy the required property. But is everything in $\mathfrak{o}(n)$ a derivative?

So given $B \in \square(n)$, with $\left.\frac{d}{dt}\right|_{t=0} e^{tB} = B$, then

we have our candidate matrix $e^{tB}$. But is it in $O(n)$?

<u>Check</u>: $\left(e^{tB}\right)\left(e^{tB}\right) = e^{tB^T} e^{tB} = e^{-tB} e^{tB} = e^0 = I$,

so $B \in \square(n) \Rightarrow e^{tB} \in O(n)$.

<u>Examples</u>: • $O(2)$. Look at $e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ = an explicit expansion

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} + \begin{pmatrix} t^2/2 & 0 \\ 0 & t^2/2! \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

• <u>$SL(n, \mathbb{R})$</u>. what is the Lie algebra?

$SL(n, \mathbb{R}) = \{ M \mid \det M = 1 \}$, we claim

$\mathfrak{sl}(n, \mathbb{R}) = \{ A \mid Tr(A) = 0 \}$.

<u>proof</u>: $\det(e^{tB}) = e^{t Tr(B)}$ is the crucial step in repeating the $O(n)$ argument. Eg.

<u>If</u> $B \in \mathfrak{sl}(n, \mathbb{R}) \Rightarrow \det(e^{tB}) = e^{t Tr(B)} = e^0 = 1$.

What about the Lie bracket?

In every case, the Lie bracket for us is the matrix commutator $AB - BA$.

## Adjoint maps.

Given $G$, we have $I(g): G \longrightarrow G \quad (g \in G)$

$$h \longmapsto ghg^{-1}.$$

Now given $g \in G$, the adjoint map is

$$\mathrm{Ad}(g): \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$B \longmapsto \frac{d}{dt}\Big|_{t=0} g e^{tB} g^{-1}$$

This gives a representation of $G$ on its Lie algebra $\mathfrak{g}$!

$\mathrm{ad}(B): \mathfrak{g} \longrightarrow \mathfrak{g}$. Given $B \in \mathfrak{g}$, we map

$$\mathfrak{g} \longrightarrow \mathfrak{g}$$

$$C \longmapsto \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(e^{tB})C.$$

<u>Theorem</u>: $\mathrm{ad}(B)(C) = [B, C]$. We can prove this:

(1) $\mathrm{Ad}(g)(C) = \frac{d}{dt}\Big|_{t=0} g e^{tC} g^{-1} = gCg^{-1}$

(2) $\mathrm{ad}(B)(C) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(e^{tB})C = \frac{d}{dt}\Big|_{t=0} e^{tB} C e^{-tB}$

$$= Be^{tB}Ce^{-tB} - e^{tB}Ce^{-tB}\Big|_{t=0}$$

$$= BC - CB.$$