Pad. Groups and Geometry.

We begin with work of F. Bachmann, and his students, and Pambuccian, Szeczebera, etc. Artzy

The idea is: If we start with an abstract group, and impose as many properties as possible that groups of isometries usually have, how much geometry can we recover from our initial abstratt group?

For example, see the "Geometry with Reflections" handout for some important properties of geometry formally encoded in a language of isometries/groups.

We associate

$$\text{point } P \longleftrightarrow \text{half-turns around } P = \mathrm{rot}(P, 180°)$$

$$\text{line } a \longleftrightarrow \text{reflection in } a,$$

note that both maps on the right are involutions.

Let's translate some geometric statements into algebra:

① P lies on a iff $\underbrace{a \circ P = P \circ a}_{\text{rotations and reflections, they commute.}}$

You can check that the abstract notion on the left agrees with our typical notion of "P lies on a".

(2) a and b are perpendicular lines iff $a \circ b = b \circ a$

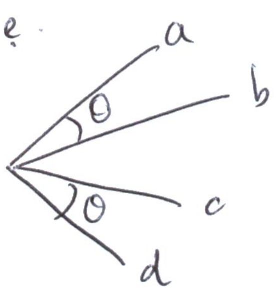
(3) a, b, c, d concurrent and $\angle(a,b) = \angle(b,c)$

or

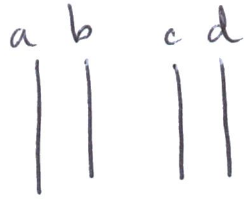
no two of a,b,c,d intersect and $\mathrm{dist}(a,b) = \mathrm{dist}(b,c)$

iff $a \circ b = d \circ c$

ie.



or



(This will allow us to capture angles and distances)

... and so on (see handout).

We can think of the entries of the table as all possible ways of filling in the equation

$$\square \circ \square = \square \circ \square$$

with a combination of points and lines in each place.

So we proceed as follows: Fix a group G, and $S \subset G$ a subset of involutions of G, ie $S = \{\alpha \mid \alpha^2 = 1\}$.

We can 'prove' things, for example:

① Recall that even in non-Euclidean geometries, every isometry is a composition of at most 3 reflections.

In general ~~Then~~ since $a \circ b =$ a translation if $a \parallel b$

$a \circ b =$ a rotation if $a \nparallel b$

we get $a \circ b \circ c =$ rot $\circ$ reflection

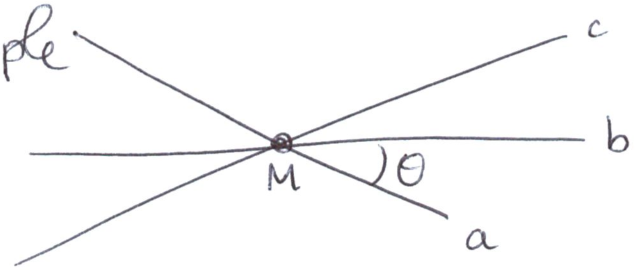
or $=$ trans $\circ$ reflection

or $=$

$=$

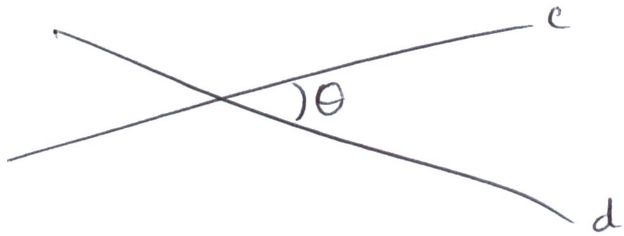
glide reflections.

For example



Then $(ab)c = \mathrm{rot}(M, 180)\, c$

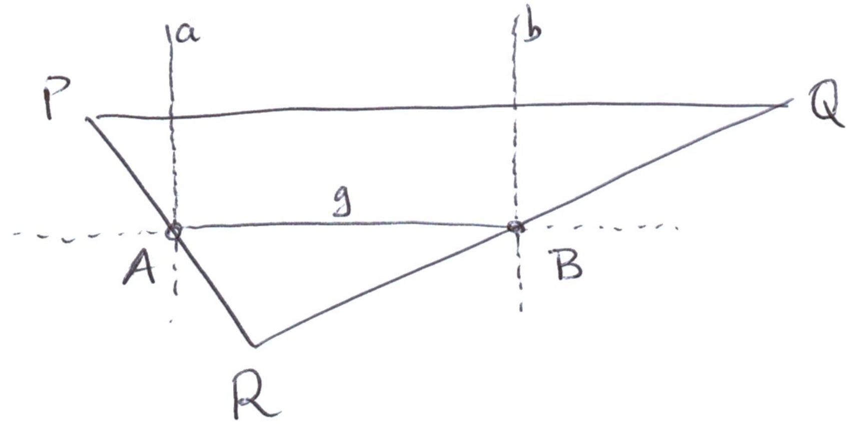
and if we choose d so that then



$\mathrm{rot}(M, 180)\, c = d \cdot d\, c$

$= c$.

② Or consider the following claim:



If A, B are midpoints then $AB \parallel PQ$. Let us name everything and 'prove' this in a group theoretic setting:

We compute $a \circ b = a \circ 1 \circ b$

$$= a \circ g \circ g \circ b$$
$$= (a \circ g)(g \circ b)$$
$$= A \cdot B$$

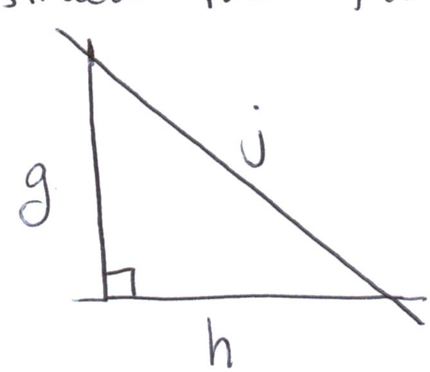
So everything we know about $a \circ b$ also says something about $A \cdot B$. So, $a \circ b = \text{trans}(2\overline{AB})$ and (it's a few steps yet) we can derive that AB and PQ are parallel.

3) Write $a | b \Leftrightarrow ab = ba$ or $a \perp b$.

Consider

$$\exists\, g, h, j \text{ s.t. } (g | h) \wedge (j \nmid g) \wedge (j \nmid gh).$$

This constructs the following picture:



So, we take these principles as axioms of a "group-based" geometry, called Tarski's axioms. We also have axioms for hyperbolic, spherical geometry in the same way.

Hyperbolic is a bit more complicated. It is possible to have $S \subset G$, S a set of involutions as before. Then we can have $a \cdot b = p \notin S$, even if $a, b \in S$.

<u>Note</u>: Given a set S, it is possible to axiomatically distinguish points from lines.