Alexander invariants of knot groups.

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- The Alexander matrix of a group.

For us, G will always be a finitely presentable group. Take

$$G = \langle x_1, \dots, x_k \mid r_1, \dots, r_\ell \rangle$$

so $G = F_k / \langle\langle r_i \rangle\rangle \cong \langle x_1, \dots, x_k \rangle / \langle\langle r_i \rangle\rangle$

Fox derivatives are a method of taking derivatives of elements of free groups:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad \frac{\partial uv}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}$$

also an inverse formula: $\dfrac{\partial u^{-1}}{\partial x_j} = -u^{-1} \dfrac{\partial u}{\partial x_j}$.

Note that the derivatives $\dfrac{\partial u}{\partial x_j}$ are not elements of the free group any more, they're elements of $\mathbb{Z}G$.

The Fox Jacobian of the finally presented group G is then

$$D = \left(\frac{\partial r_i}{\partial x_j} \right)$$

Let $\alpha : G \longrightarrow \mathbb{Z}^{\langle t \rangle}$ be the abelianization, we're assuming $G/G' \cong \mathbb{Z}$ for simplicity.

Then α extends to a map of group rings

$\alpha : \mathbb{Z}G \longrightarrow \mathbb{Z}[t^{\pm 1}]$ group ring of integers.

Then the Alexander matrix is

$$A = \left(\tilde{\alpha}\left(\frac{\partial r_i}{\partial x_j}\right)\right)$$ (ie, you apply the Abelianization to all entries of D).

The elementary ideals $\mathcal{E}_\ell$ of $\mathbb{Z}[t^{\pm 1}]$ are generated by the $(k-\ell)\times(k-\ell)$ minors of A. Ideals are still hard to work with, ideally we want a single polynomial to work with.

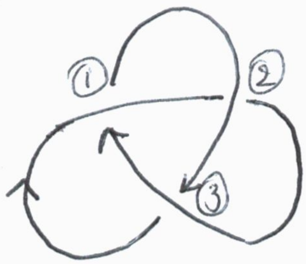
To get a polynomial, we take the smallest principal ideal containing $\mathcal{E}_m$. Its generator is the polynomial we want, though it is only defined up to units.

Knot groups.

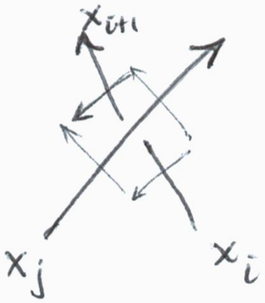
Take a knot $K \subset S^3$, then $G_K = \pi_1(S^3 - \nu(K))$ (a neighbourhood of K is $\nu(K)$)

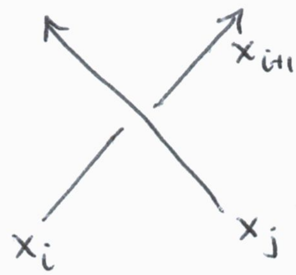
We calculate G_K via the Wirtinger presentation.

<u>Ex</u>: Trefoil:



Then at each crossing we have a rule





which gives:

$$x_{i+1} = x_j^{-1} x_i x_j \quad \text{and} \quad x_{i+1} = x_j x_i x_j^{-1}$$

$$\text{or } 1 = x_{i+1}^{-1} x_j^{-1} x_i x_j = r_i \quad \text{or } 1 = x_j x_i x_j^{-1} x_{i+1}^{-1} = r_i$$

For the trefoil,
$$r_1 = x_3^{-1} x_1^{+1} x_3 x_2^{-1}$$
$$r_2 = x_1^{-1} x_2 x_1 x_3^{-1}$$
$$r_3 = x_2^{-1} x_3 x_2 x_1^{-1}.$$

Then we compute Fox derivatives:

$$\frac{\partial r_1}{\partial x_1} = x_3^{-1}, \quad \frac{\partial r_1}{\partial x_2} = -x_3^{-1} x_1 x_3 x_2^{-1} \quad \frac{\partial r_1}{\partial x_3} = -x_3^{-1} + x_3^{-1} x_1$$

$$\frac{\partial r_2}{\partial x_1} = -x_1^{-1} + x_1^{-1} x_2, \quad \frac{\partial r_2}{\partial x_2} = x_1^{-1} \quad \frac{\partial r_2}{\partial x_3} = -x_1^{-1} x_2 x_1 x_3^{-1}$$

$$\frac{\partial r_3}{\partial x_1} = -x_2^{-1}x_3x_2x_1^{-1}, \quad \frac{\partial r_3}{\partial x_2} = -x_2^{-1} + x_2^{-1}x_3, \quad \frac{\partial r_3}{\partial x_3} = x_2^{-1}.$$

In the case of knot groups, $\alpha(x_i) = t$ for all i. So we get an Alexander matrix of

$$A = \begin{pmatrix} t^{-1} & -1 & -t^{-1}+1 \\ -t^{-1}+1 & t^{-1} & -1 \\ -1 & -t^{-1}+1 & t^{-1} \end{pmatrix}$$

<u>Claim</u>: $\mathcal{E}_0 = \det A = 0$. We can see this since the columns c_1, c_2, c_3 satisfy $-c_1 - c_2 = c_3$.

This happens in general and follows from the fundamental identity of Fox derivatives:

$$\sum_i \left(\frac{\partial r_i}{\partial x_j}\right)\Big|_{x_i=t} (1-t) = 0.$$

- This is equivalent to A having a right eigenvector with eigenvalue 0.
- The matrix A also has a left eigenvector, and it's always $[1, 1, \dots 1]$ (all ones). So it has eigenvalue zero.

So we can do row/column operations to get

$$A \rightsquigarrow \begin{bmatrix} t^{-1} & -1 & 0 \\ -t^{-1}+1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we can conclude that $\mathcal{E}_1$ is principal, as only one 2×2 determinant is nonzero.

We get the determinant $\Delta_K(t) = t^{-1} - 1 + t$, the Alexander polynomial.