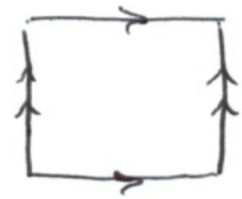
Intro to knot theory part 2.

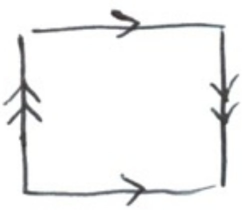
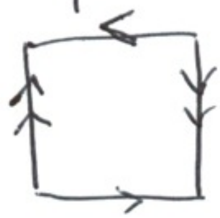
Seifert Surfaces.

By a surface Σ we mean a metric space such that every $x \in \Sigma$ has a neighbourhood U such that either

$U \cong \mathbb{R}^2$ or $U \cong \mathbb{R}^2_+ = \{(x,y) \mid y \geq 0\}$.

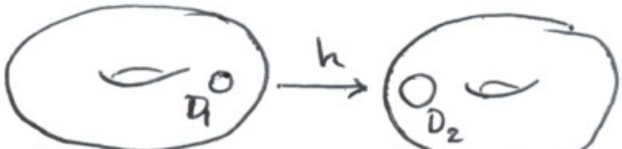
Examples:

Without boundary: S^2  T^2  = 

K^2  , P^2  ; with boundary 

The connected sum of two surfaces Σ_1 and Σ_2 is obtained by removing open disks $D_i \subset \mathrm{int}(\Sigma_i)$, $i=1,2$, and gluing the resulting boundaries via a homeomorphism h.

E.g. Σ_1 Σ_2 Σ_3



Notation: $\Sigma_3 = \Sigma_1 \# \Sigma_2$.

Theorem: Every surface without boundary is homeomorphic to one of the following: $S^2, T^2, T^2 \# T^2, T^2 \# T^2 \# T^2, \ldots$

or $P^2, P^2 \# P^2, P^2 \# P^2 \# P^2, \ldots$.

Every surface with boundary is homeomorphic to one of the surfaces above, with some open disks removed.

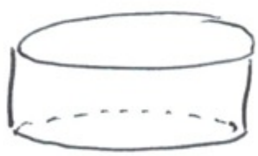
Thus we have two kinds of surfaces:

Orientable: S^2 and connect sums of tori (perhaps with disks removed)

Non-orientable: Connect sums of P^2.

Conceptually, orientable means "two-sided" and non-orientable "one-sided".

E.g: Orientable



Non-orientable

Restrict our attention to orientable surfaces: If $\partial\Sigma = \emptyset$, then

Def: The genus of a surface $\Sigma \neq S^2$ is the number of tori in its connect sum decomposition. Write

$$g(\Sigma) = g(\underbrace{T^2 \# \cdots \# T^2}_{k \text{ tori}}) = k.$$

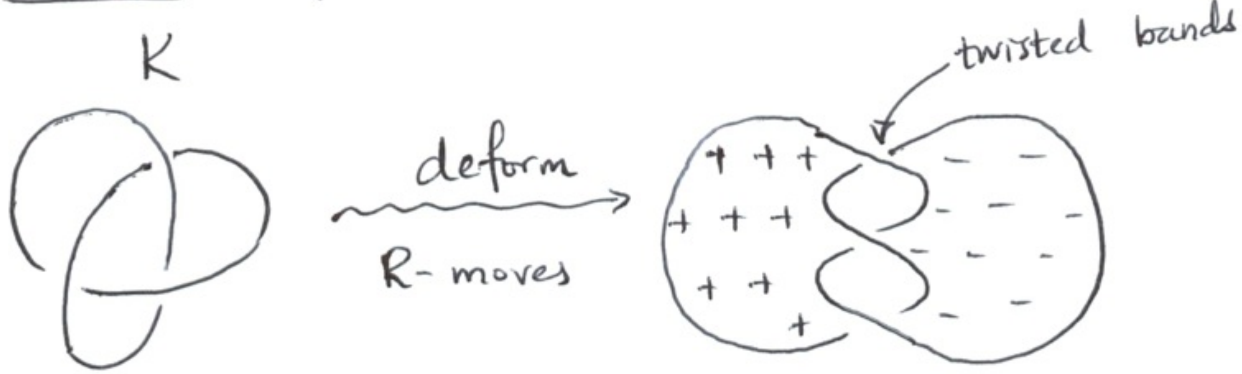
Set $g(S^2) = 0$.

If $\partial\Sigma \neq \emptyset$, then upon filling $\partial\Sigma$ with disks $D_1^2, \ldots, D_n^2$, we get a connect sum of tori.

Define
$$g(\Sigma) = g\left(\Sigma \cup \left(\dot{\bigcup_{i=1}^{n}} D_i^2\right)\right) = g(\underbrace{T^2 \# \ldots \# T^2}_{k \text{ tori}}) = k.$$

Definition: Given a knot $K \subset S^3$, a Seifert surface for K is an orientable surface $S \subset S^3 \setminus K$ with $\partial S = K$.

Example: If K is the trefoil, then



Here, '+' and '−' are meant to be colourings of the 'top' and 'botton' of a two-sided surface.

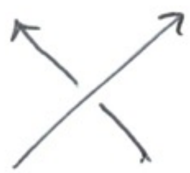
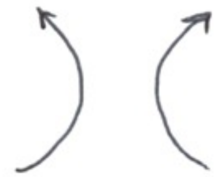
Theorem (Seifert, 1934):

Every $K \subset S^3$ admits a Seifert surface.

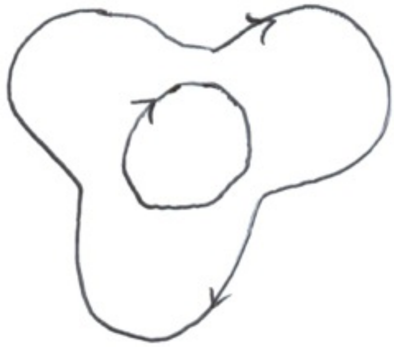
Proof: Seifert's algorithm:

1. Given a diagram of K, introduce an orientation. (Thinking of $D \subset \mathbb{R}^2 \subset \mathbb{R}^3$).

2. Smooth all crossings, ie make local changes

and  becomes 

So we get

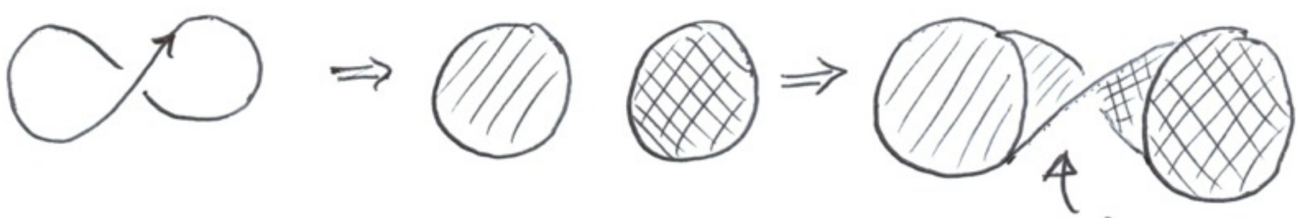


The result of smoothing every crossing is a set of disjoint, possibly nested circles in the plane. These are <u>Seifert circles</u>

(3) Move nested circles out of the plane in $\mathbb{R}^3$, and fill all circles with disjoint disks.
(Raise middle disk out of the board).

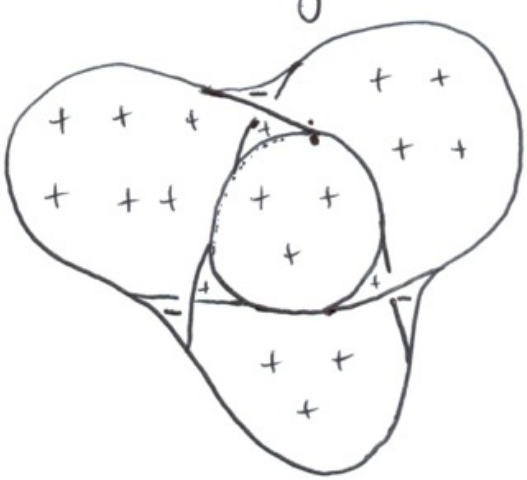
(4) "Un-smooth" crossings by adding twisted bands where each crossing was before.

<u>Eg.</u>



surface 'flips' here.

For the trefoil, we get:



Def: The genus $g(K)$ of a knot is a knot invariant given by $g(K) = \min\{g(S) \mid S \text{ a Seifert surface for } K\}$.

In practice, this is hard to calculate. However there are tricks for some knots.

Def: A diagram D is alternating if, when oriented, one alternately encounters under crossings and over crossings when traveling along the knot.

E.g. Trefoil, or

is alternating.

A knot is alternating if it admits an alternating diagram.

Theorem (??): If K is an alternating knot, then applying Seifert's algorithm to an alternating diagram of K yields a minimal genus Seifert surface.

Example: The trefoil K.

The Euler characteristic of a surface is

$$\chi(S) = V - E + F,$$

where V, E, F are the number of $0, 1, 2$ cells in a CW decomposition of S.

We also have

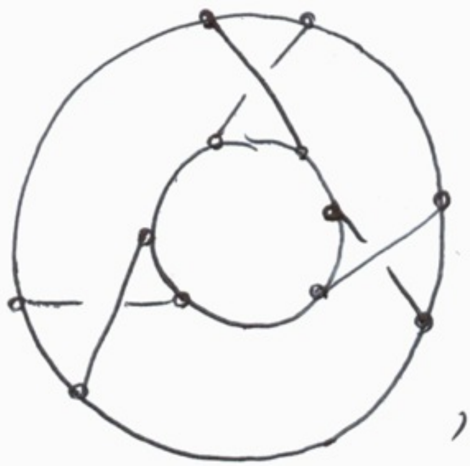
$$\chi(S) = 2 - 2g(S) - |\partial S|$$

but in our case, since $\partial S = K$, $|\partial S| = 1$ and

$$\chi(S) = 1 - 2g(S).$$

Thus $V - E + F = 1 - 2g(S) \Rightarrow g(S) = \dfrac{1 - V + E - F}{2}$.

For the trefoil, Seifert's algorithm gave: (Note  alternates)



$V = 12$

$E = 18$

$F = 5$

, so $g(S) = \dfrac{1 - 12 + 18 - 5}{2} = 1$

Thus the genus of the trefoil is 1.

Theorem: If K admits an alternating diagram with c crossings which yields s Seifert circles upon applying Seifert's algorithm, then

$$g(K) = \frac{1 + c - s}{2}.$$

Example:

smooth $\rightsquigarrow$

3 Seifert circles

$g(K) = \dfrac{1 + 4 - 3}{2} = 1$, or genus of unknot is 0.

<u>Proposition</u>: A knot has genus zero if and only if it is the unknot.

<u>Proof</u>: A genus 0 surface with one boundary component is a disk. If K bounds a disk, D, then a tiny nbhd at the centre of D is essentially a standardly embedded disk D' with $\partial D' \simeq K$.

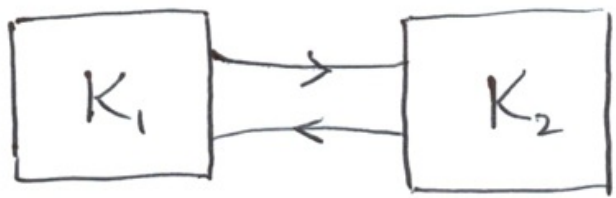
Application of genus:

Define the knot sum of knots K_1 and K_2 via the diagram:

If

then

$K_1 \# K_2$ is



A knot K is called prime if $K \neq K_1 \# K_2$ for some pair of non-trivial knots K_1 and K_2.

Proposition: $g(K_1 \# K_2) = g(K_1) + g(K_2)$.

Proof: Use the Seifert circles formula.

Corollary: Any knot with genus 1 is prime.

Proof: If $g(K) = 1$ and $K = K_1 \# K_2$, then $1 = g(K_1) + g(K_2)$ implies either $g(K_1) = 0$ or $g(K_2) = 0$. Thus one of the summands is trivial.