Intro to Classical Knot Theory.

(Based on minicourse notes of Roger Fenn).

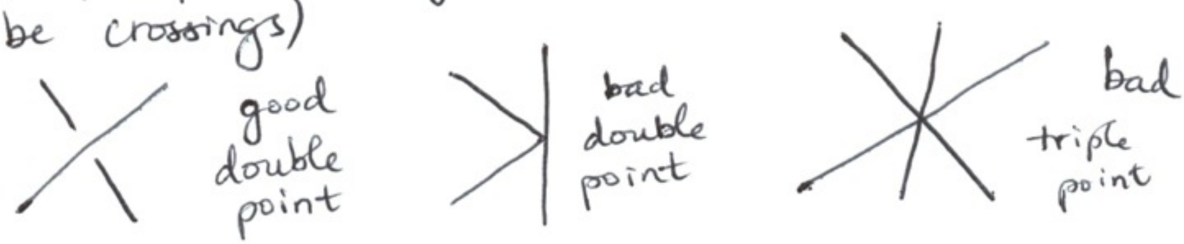
Def: Suppose we have two embeddings $k_i : S^1 \hookrightarrow \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} \simeq S^3$. Two such embeddings k_1, k_2 are called equivalent if there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{tikzcd} \mathbb{R}^3 \arrow[rr, "h"] & & \mathbb{R}^3 \\ & S^1 \arrow[ul, "k_1"] \arrow[ur, "k_2"'] & \end{tikzcd}$$

commutes. The equivalence classes under this relation are <u>knots</u>. We'll often identify a class with a specific representative K.

If K is polygonal or smooth, then K is <u>tame</u>. Knots that are not tame are <u>wild</u>, we won't consider wild knots.

Def: A <u>diagram</u> of a knot is an immersion of S^1 in $\mathbb{R}^2$ via composition with projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, which is in 'general position', together with crossing information at every double point.
(I.e. no triple points or greater allowed, double points must be crossings)



Eg: Trefoil

Figure 8

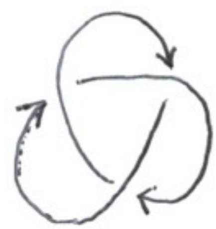
What of orientation?

We could orient K, then if

K is  then $-K$ is 

(reverse the arrows)

we could also reverse the orientation of the ambient space by composing with $f(x,y,z) = (x,y,-z)$. Then

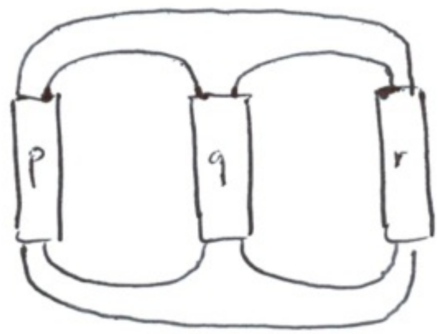
$\overline{K} =$  and $-\overline{K}$ is 

all of these may be inequivalent, so we need to keep track of orientations carefully. (Orientation-preserving equivalence now)

- If $K \cong \overline{K}$, then K is called amphichiral.

  The trefoil is <u>not</u> amphichiral because K and $\overline{K}$ are different (hard to show).

• If $K \cong -K$, then K is invertible. The simplest non-invertible knot is



 right hand twist.

with $(p,q,r) = (3,5,7)$ right hand twists.

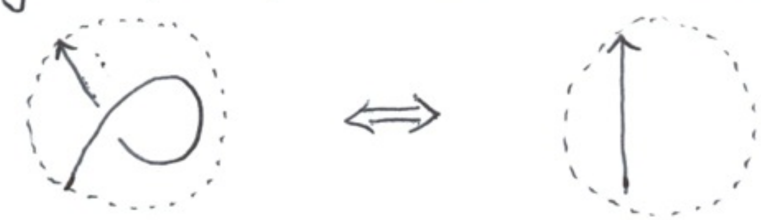
equal

Fundamental question: When are two knots equivalent?

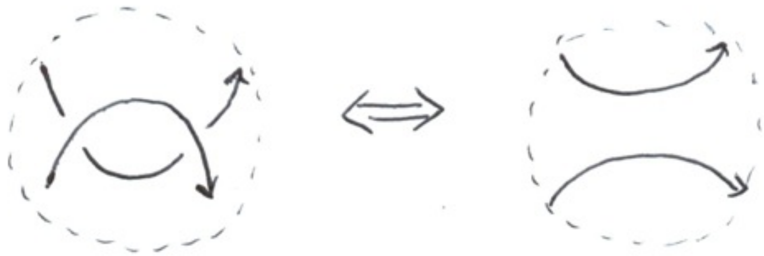
I.e. when are $k_i : S^1 \longrightarrow \mathbb{R}^3$, $i=1,2$ members of the same equivalence class? (We ask for $h: \mathbb{R}^3 \to \mathbb{R}^3$ orientation-preserving)

Ans: If the diagram for K_1 can be changed to the diagram for k_2 via Reidemeister moves: (unoriented) or oriented

RI:



RII



and

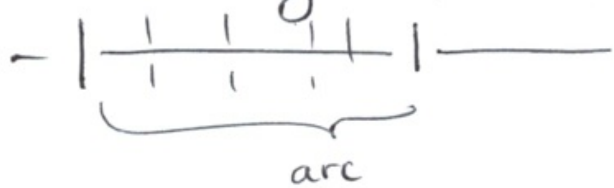
RIII

Of course, if there exists a sequence of Reidemeister moves it may be nearly impossible to find.

<u>Theorem</u>: Let D be a diagram of the unknot with c crossings. Then there is a sequence of at most $(231c)^{11}$ Reidemeister moves transforming D into $\bigcirc$.

<u>Def</u>: A <u>knot invariant</u> is any "quantity" one assigns to knot diagrams that is invariant under R-moves.

<u>Example</u>: An <u>arc</u> of a diagram D is a piece going from one undercrossing to another



Suppose that a diagram D can be coloured with red, white and yellow so that no crossing has two colours. Then call D 3-colourable.

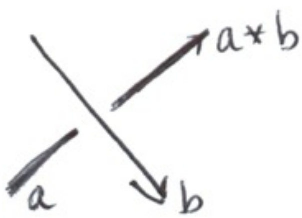
<u>Fact</u>: 3-colourability is a knot invariant. Trefoil, fig 8, $\bigcirc$.

Here's how to check. Suppose we have

$X = \{\text{set of colours}\}$

and the arcs of D are coloured by elements of X.

Define $\forall a, b \in X$ used to colour D an operation $a * b$ by



note orientations!

RHR to determine $a*b$.

So 3-colourability gives $X = \{R, W, Y\}$ and "no crossing has two colours" forces multiplication table

$*$	R	Y	W
R	R	W	Y
Y	W	Y	R
W	Y	R	W.

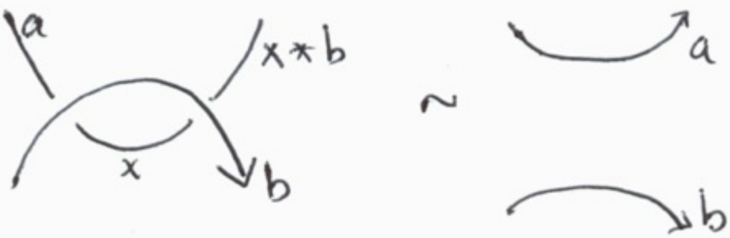
Asking for invariance of $(X, *)$ under Reidemeister moves then amounts to:

<u>RI</u>.

$a * a = a.$ (i)

<u>RII</u>

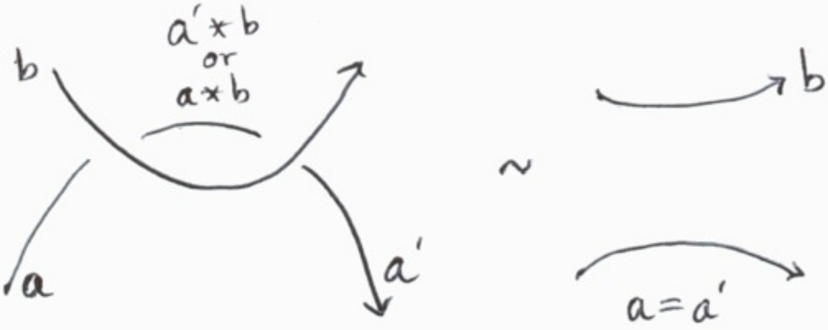
(ii) a)



going $\Leftarrow$, $\exists\, x$ s.t. $x * b = a$

$- * b$ surjective.

or the other RII:



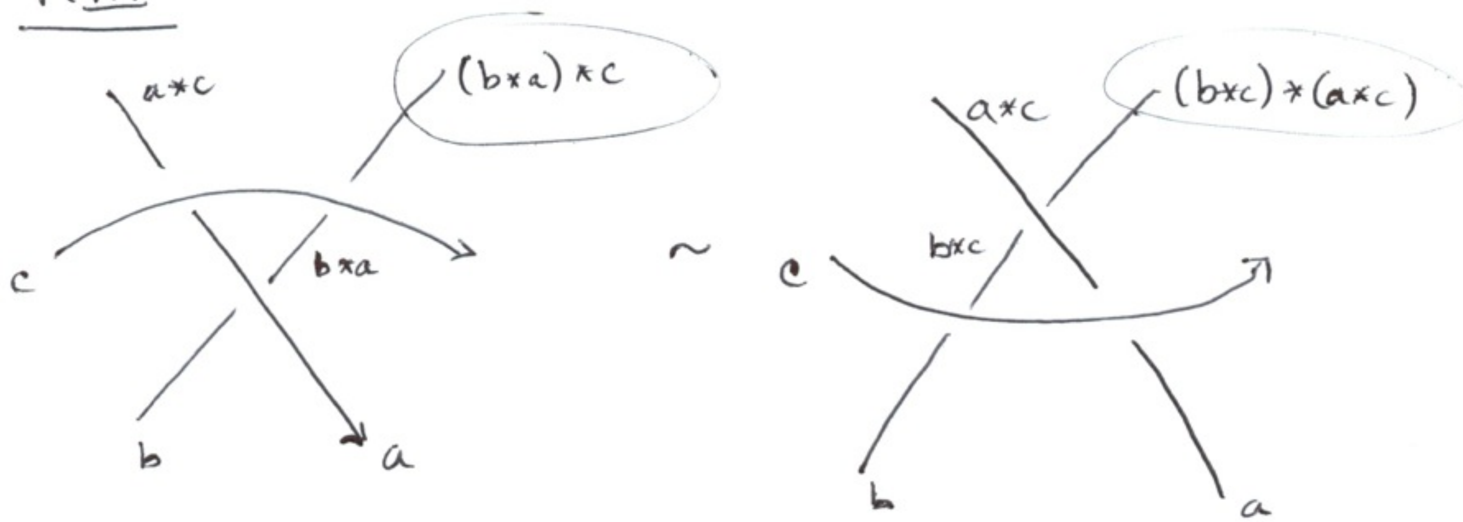
(ii)(b)

$a' * b = a * b$

$\Rightarrow a = a'$

$\Rightarrow - * b$ injective.

RIII



So we need $(b*a)*c = (b*c)*(a*c)$. (iii)

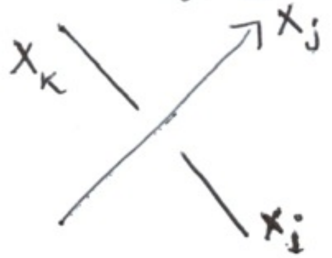
Any X with $*$ satisfying (i)—(iii) is called a quandle. Colourings of arcs by quandle elements is a knot invariant.

Example: $X = \{R, W, Y\}$ with multiplication table as before is a quandle. Therefore 3-colouring is a knot invariant.

Example: If G is a group and $X \subset G$, and $gXg^{-1} = X$ $\forall g \in G$, then

$$g*h = h^{-1}gh \quad \text{defines a quandle}$$

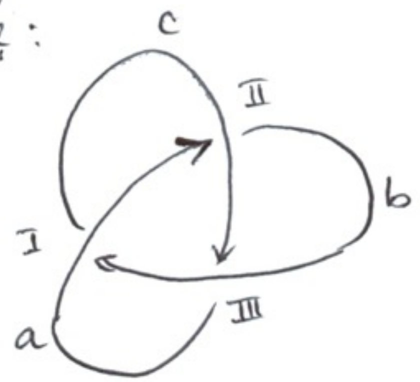
Example: Given a diagram D, label the arcs $\{x_1, \dots, x_n\} = X$, and consider the free group $F(X)$. Mod out by relations arising from conjugation as the quandle action, i.e.



gives $x_i * x_j = x_j x_i x_j^{-1} = x_k$.

Then the generators of the resulting group $F(X)/R$ form a quandle, and in fact $F(X)/R$ is well known. Given $K \subset \mathbb{R}^3$, $F(X)/R$ constructed as above is $\pi_1(\mathbb{R}^3 \setminus K) \simeq F(X)/R$, the fundamental group of the complement.

<u>Example</u>:



$\Rightarrow$ $X = \{a, b, c\}$ and relations

I. $c = b * a = aba^{-1}$

II. $b = a * c = cac^{-1}$

III. $a = c * b = bcb^{-1}$

It so happens one relation is always redundant, and we get (upon eliminating c from II).

$b = aba^{-1} \cdot a \cdot ab^{-1}a^{-1}$

$\Rightarrow b = abab^{-1}a^{-1}$ or $bab = aba$.

So the group is

$$\langle a, b \mid bab = aba \rangle \simeq B_3.$$