The Hopf invariant (Derek Krepski).

Consider $h : S^3 \longrightarrow S^2$, where $S^2 = \mathbb{C}P^1 \simeq \left(\mathbb{C}^2 \setminus \{0\}\right) / \mathbb{C}^\times$

and $S^3 \subseteq \mathbb{C}^2$. Define

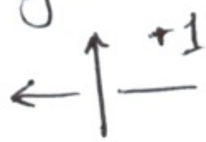
$h(z_0, z_1) = [z_0 : z_1]$, here we're using homogeneous coordinates.

Consider $K_1 = h^{-1}[0 : 1]$, and $K_2 = h^{-1}[1 : 0]$. What is their linking number, $\mathrm{link}(K_1, K_2)$?

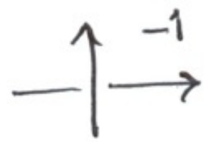
Using diagrams,

$$\mathrm{link}(K_1, K_2) = \frac{1}{2} \sum_{\substack{\text{“}K_1 \cap K_2\text{”},\\ \text{thought of}\\ \text{as diagram crossings}}} \pm 1$$

where each crossing is assigned a ± 1 depending on some rule. Usually by assigning orientations and using the right hand rule on some diagrams:



and

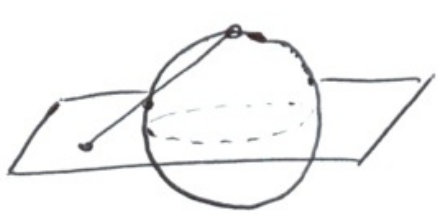


So we use stereographic projection on our K_1 and K_2:

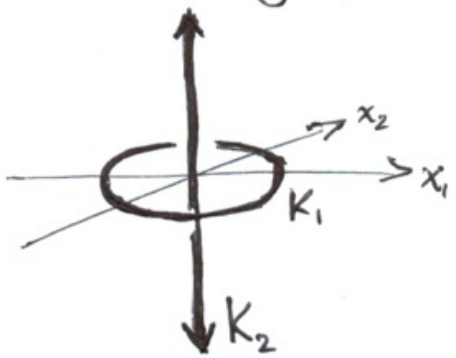
$K_1 = \{(0, z_1) \mid |z_1| = 1\}$, $K_2 = \{(z_0, 1) \mid |z_0| = 1\}$.

Then the projection is

$$\mathrm{proj} : (x_1, x_2, x_3, x_4) \longmapsto \left(\frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4}\right)$$

thinking , as usual.

Then the image of K_2 under proj is a circle in the $x_1 x_2$ plane, and the image of K_1 is the x_3-axis, thinking of $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$:



Another definition of the Hopf invariant:
Thinking of the map $h: S^3 \longrightarrow S^2$ again, use it as an attaching map $\mathbb{C}P^2 = S^2 \cup_h D^4$

So $\mathbb{C}P^2$ has cells in dimension 2 and 4, and a point in dimension 0 if you like. Because there are no odd-dimensional cells

$$H^*(\mathbb{C}P^2) = \begin{cases} \mathbb{Z} & \text{if } * = 4 \\ 0 & \text{if } * = 3 \\ \mathbb{Z} & \text{if } * = 2 \\ 0 & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \end{cases}$$

Then the product structure gives

$$H^1(\mathbb{C}P^2) \simeq \langle 1 \rangle, \quad H^2(\mathbb{C}P^2) \simeq \langle a \rangle, \quad H^4(\mathbb{C}P^2) \simeq \langle a^2 \rangle$$

For a general map $f : S^3 \to S^2$, you can still construct $X = S^2 \cup_f D^4$ and get

$$H^*(X) = \begin{cases} \mathbb{Z} \simeq \langle b \rangle \\ 0 \\ \mathbb{Z} \simeq \langle a \rangle \\ 0 \\ \mathbb{Z} \simeq \langle 1 \rangle \end{cases}$$

Then one can show that the product structure on the cohomology ring satisfies $a^2 = H(f)\, b$ for some integer $H(f)$. This integer is the Hopf invariant.

More generally, consider $f : S^{2n-1} \longrightarrow S^n$ and $X = S^n \cup_f D^{2n}$. Then $H^*(X)$ is concentrated in dim. 0, n and $2n$; this allows us to define the Hopf invariant using $a^2 = H(f)\, b$.

Question: Fixing $n \geq 2$, what are the possible values of $H(f)$ for different maps f?

Difficult refinement of this question: For which n does $\exists\, f : S^{2n-1} \longrightarrow S^n$ with $H(f) = \pm 1$?

Answer: $n = 2, 4, 8$ (due to Frank Adams and Atiyah) (maybe others).

Let's understand the consequences of this answer.

Application:

The solution to the Hopf invariant 1 problem settles:
For which n does $\mathbb{R}^n$ have a bilinear multiplication with no zero divisors:

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Note that for $n = 1, 2, 4, 8$ we have the real numbers, complex numbers, quaternions and octonions.

How does the Hopf invariant relate to this?

Idea: Given such a structure, we'll construct a map of Hopf invariant 1, using the resulting "multiplication" on $S^{n-1} \subseteq \mathbb{R}^n$.

Def: X is an H-space if $\exists$ a continuous map $\mu: X \times X \longrightarrow X$ with $e \in X$ satisfying $\mu(x, e) = \mu(e, x) = x$.
(Note: Sometimes this equality is only required "up to homotopy")

So given a multiplication on $\mathbb{R}^n$, we get

$$S^{n-1} \times S^{n-1} \longmapsto S^{n-1}$$
$$(x, y) \longmapsto \frac{xy}{|xy|} \in S^{n-1}.$$

Not quite a map of spheres as we want, but we'll eventually get to a map $S^{2n-1} \longrightarrow S^n$.

Def: Given X, ~~SX~~ SX (suspension of X) is

$$X \times [0,1] / \sim, \quad \text{where } \begin{array}{l}(x,0) \sim (y,0) \\ (x,1) \sim (y,1)\end{array} \ \forall x, y \in X.$$

Def: Given X, Y, the join $X * Y$ is

$$X \times [0,1] \times Y /\sim \quad \text{, where } (x,0,y) \sim (x',0,y)$$
$$(x,1,y) \sim (x,1,y').$$

E.g. $S^1 * S^1 \cong S^3$, a homeomorphism is

$$[x,t,y] \longmapsto \left(\sin\left(\tfrac{\pi t}{2}\right)x, \cos\left(\tfrac{\pi t}{2}\right)y\right)$$

thinking of x and y as unit complex numbers.

In general, $S^n * S^m \cong S^{n+m+1}$.

<u>The Hopf construction:</u>

Given $f: X \times Y \longrightarrow Z$, there is a map $H_f: X * Y \longrightarrow SZ$ whose formula is $H_f([x,t,y]) = [f(x,y),t]$.

Applied to the multiplication map, this

$$\mu: S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$$

gives a map

$$H_\mu: S^{n-1} * S^{n-1} \cong S^{2n-1} \longrightarrow SS^{n-1} \cong S^n.$$

This map has Hopf invariant 1. So we showed that

$$\begin{array}{c}\text{multiplication} \\ \text{on } \mathbb{R}^{2n}\end{array} \implies \begin{array}{c}\text{a Hopf invariant} \\ 1 \text{ map on } \cancel{\mathbb{R}}\, S^{2n-1}\end{array}$$