The braid groups B_n

Four definitions:
- As strings
- As generators with relations
- As mapping class groups
- As fundamental groups.

+ "main theorem" for each approach.

Def 1

Let $D \subset \mathbb{R}^2 \cong \mathbb{C}$ be the closed unit disk, and let $\{p_1, \dots, p_n\}$ be evenly-spaced points on the x-axis.

Suppose for $i = 1, \dots, n$ you have n paths $\beta_i : [0,1] \longrightarrow D \times [0,1]$ satisfying $\beta_i(0) = (p_i, 0)$ and $\beta_i(1) = (p_j, 1)$, and each path intersects each disc $D \times \{t\}$ exactly once, and is the identity in the second coordinate.

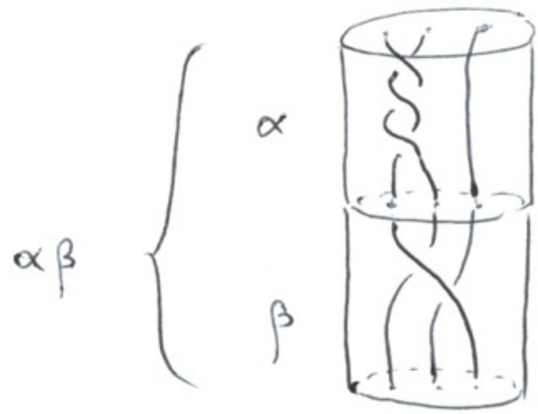
E.g.

An n-braid is n such paths

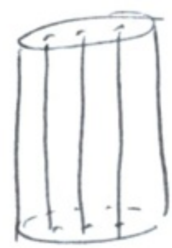
$$\beta(t) = (\beta_1(t), \beta_2(t), \dots, \beta_n(t))$$

with $\beta_i(t) \neq \beta_j(t)$ whenever $i \neq j$.

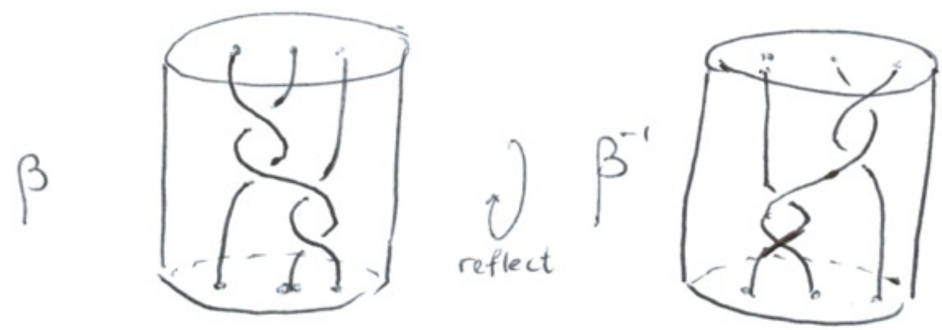
Define an equivalence relation on n-braids by
$\beta_1 \sim \beta_2$ if one can be deformed into another by a sequence of
n-braids. (Isotopy). (Ambient isotopy).
Then B_n is the set of equivalence classes, with concatenation
as operation:



Identity in B_n is vertical lines



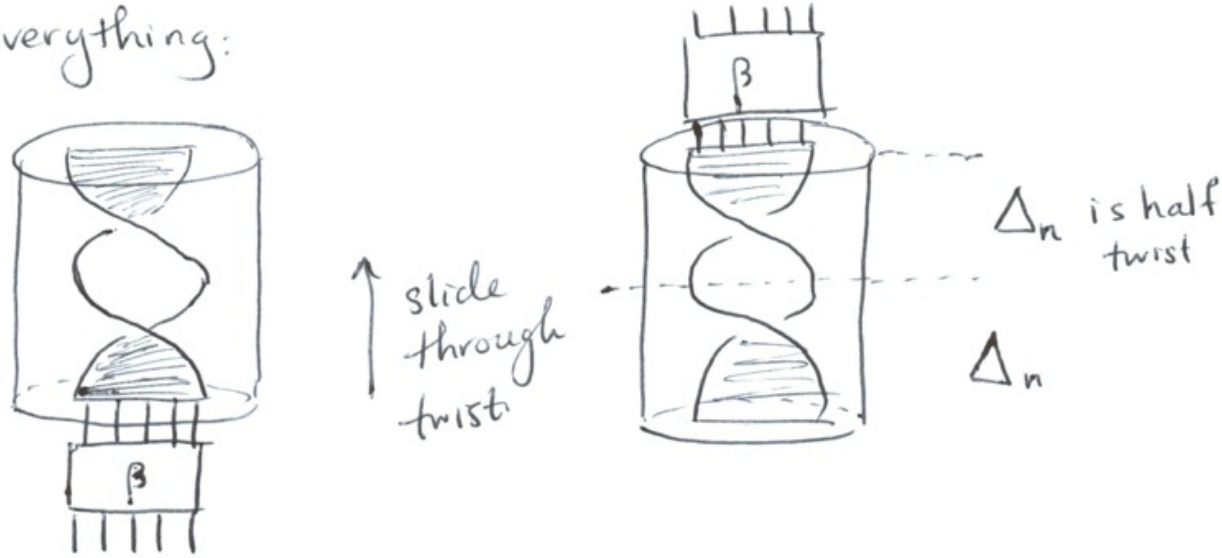
Inverses are given by vertical reflection.



Then stacking β and β^{-1} gives, up to ambient
isotopy, straight lines.

This allows for geometric understanding of algebraic properties,

<u>Example</u>: The "full twist" of all strand commutes with everything:



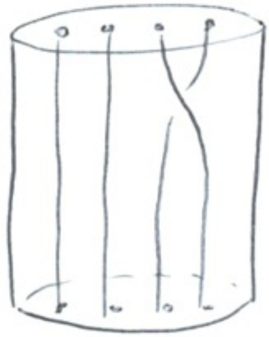
So Δ_n^2 is central in B_n, in fact $Z(B_n) = \langle \Delta_n^2 \rangle \simeq \mathbb{Z}$.

<u>Def 2</u> The group B_n is also given by the presentation:

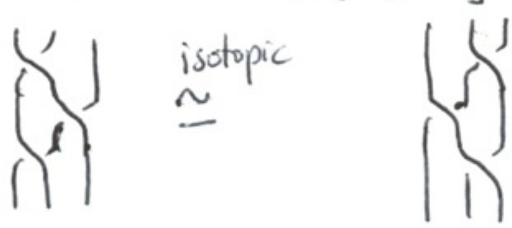
$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geqslant 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_i \text{ if } |i-j| = 1 \end{array} \right\rangle$$

where σ_i is the half-twist on strands i and $i+1$:

σ_3 in B_4



<u>Example</u>: The relation $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ says:



Main application:

Every $\beta \in B_n$ admits an expression of the form $\beta = \Delta_n^r \beta'$, where β' has only positive powers of the generators σ_i. Moreover, we can choose r and β' so that:

- $r \in \mathbb{Z}$ is maximal.
- with r maximal, amongst possible β' pick the one with generators σ_i appearing with "lexicographically minimal" subscripts.

(E.g. between $\beta' = \sigma_3\sigma_1\sigma_2\sigma_1$ and $\beta' = \sigma_3\sigma_2\sigma_1\sigma_2$, choose $\sigma_3\sigma_1\sigma_2\sigma_1$)

Then $\beta = \Delta_n^r \beta'$ is in Garside normal form.

Applications are endless

- $Z(B_n) = \langle \Delta_n^2 \rangle$, B_n is torsion free, solve word/conj. problems.
- Cryptosystems are based on Garside form, etc.

Def:

Define the mapping class group of a compact, oriented, connected, punctured surface Σ_i as follows:

$\Sigma_i =$ 

Let $P = \{p_1, \dots, p_n\}$ denote the punctures, and set

$$\mathrm{Homeo}_+(\Sigma_i, P) = \left\{ \begin{array}{l} \text{orientation-preserving homeomorphisms } h: \Sigma_i \to \Sigma_i \\ \text{with } h(P) = P \text{ and } h|_{\partial\Sigma_i} = id \end{array} \right\}$$

Then $\mathrm{Mod}(\Sigma_i, P)$ is the set of isotopy classes of such homeomorphism, with composition as group operation.

$$\mathrm{Mod}(\Sigma_i, P) = \pi_0(\mathrm{Homeo}_+(\Sigma_i, P)).$$

Example (Alexander). Let $D \subset \mathbb{C}$ be the unit disk. Then $\mathrm{Mod}(D, \emptyset)$ is trivial.

Proof: Let $h: D \to D$ be a homeomorphism with $h|_{\partial D} = \mathrm{id}$. Define $H: D \times [0,1] \to D$ by

$$H(z,t) = \begin{cases} (1-t)\, h\left(\frac{z}{1-t}\right) & \text{if } 0 \le |z| < 1-t \\ z & \text{if } 1-t \le |z| \le 1. \end{cases}$$

and let $H(z,1): D \to D$ be the identity. I.e. at time t do h on a small disk of radius $1-t$, and identity elsewhere. This deforms any h into id, so $\mathrm{Mod}(D, \emptyset) = \{1\}$.

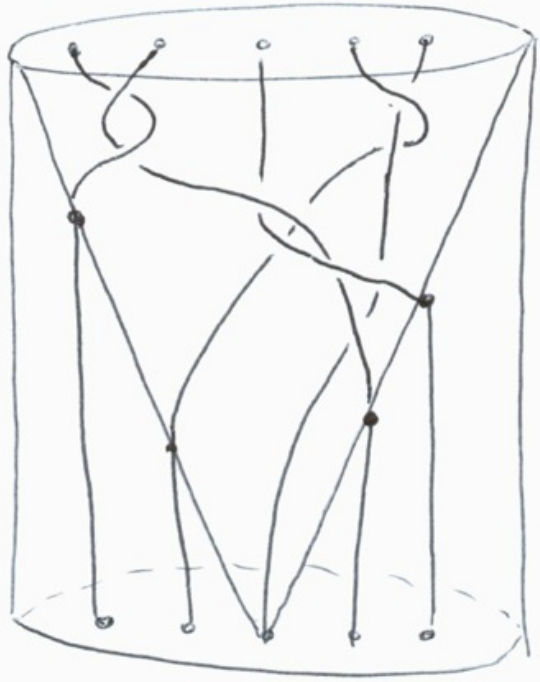
Now we can show $\mathrm{Mod}(D,P) \cong B_n$ if $P = \{p_1, \dots, p_n\}$ are evenly spaced points in D along the real axis.

Given $[h] \in \mathrm{Mod}(D,P)$, we can think of h as a representative of $[h] \in \mathrm{Mod}(D, \emptyset)$ by disregarding P. By Alexander, $[h] = 1$ if considered as an element of $\mathrm{Mod}(D,\emptyset)$, so $H(z,t)$ satisfies $H(z,0) = h(z)$ and $H(z,1) = \mathrm{id}$.

For $i = 1, \dots, n$ set $\beta_i(t) = H(p_i, t)$. I.e, the ith strand of $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ is the path traced by $h(p_i)$ as we deform $h(D)$ to the identity.

We get:

$H(z,0)$



$H(z,1)$.

This defines a map

$$\mathrm{Mod}(D,P) \longrightarrow B_n,$$

and it is in fact an isomorphism.

Sample application: The surface $D \setminus P$ admits a hyperbolic metric, so B_n acts on a hyperbolic surface. Can use this to create an action of B_n on $\partial(\widetilde{D\setminus P})$ and get: There exists an embedding:

$$B \hookrightarrow \mathrm{Homeo}_+(\mathbb{R}).$$

Corollary: B_n is torsion-free, and left-orderable. I.e. there exists an ordering $<$ of B_n with $g<h \Rightarrow fg<fh \quad \forall f,g,h \in B_n$.

Def 4:

Consider the 'big diagonal'

$$\Delta = \{ (z_1, \dots, z_n) \mid z_i = z_j \text{ for some } i<j \} \subset \mathbb{C}^n.$$

Then using the basepoint $P = (p_1, \dots, p_n)$, where $p_1, \dots, p_n$ are evenly spaced along the real axis, we consider

$$P_n = \pi_1(\mathbb{C}^n \setminus \Delta).$$

In this group, an element is $\gamma : [0,1] \longrightarrow \mathbb{C}^n \setminus \Delta$, which is a choice of n non-intersection paths:

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$$

satisfying $\gamma_i(0) = p_i$ and $\gamma_i(1) = p_i$. So each element is a braid whose strands start and end at the same point.

To remove this restriction, we mod out by the action of S_n (symmetric group) on $\mathbb{C}^n \setminus \Delta$ (permuting coords)

$$B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/S^n)$$

Now $(\mathbb{C}^n \setminus \Delta)/S^n$ is a space of unordered tuples, so we no longer have strands beginning/ending at the same point.

Main application: The spaces $\mathbb{C}^n \setminus \Delta$ and $(\mathbb{C}^n \setminus \Delta)/S^n$ have trivial higher homotopy, so they are $K(P_n, 1)$ and $K(B_n, 1)$ respectively. So we can use them for computing the group cohomology of P_n & B_n.

Theorem: P_n & B_n have finitely many nontrivial cohomology groups.

Cor: They're torsion free.