LSGT – Feb 10

The Schur–Horn theorem & symplectic geometry

<u>Ref</u>: A. Knutson, "The symplectic and algebraic geometry of Horn's problem." (on arXiv)

<u>Recall</u>: If A is self-adjoint (Hermitian), ie. $A^* = A$ ← conjugate, transpose

- A has real eigenvalues
- A is unitarily diagonalizable
  ie. $\exists$ unitary matrix $P \in U(n)$ $\quad (PP^* = id)$
  w/ $P^*AP = D$ ← diagonal matrix.
- $\mathrm{diag}(A) := (a_{11}, a_{22}, \ldots, a_{nn}) \in \mathbb{R}^n$

For given $\lambda \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$
let

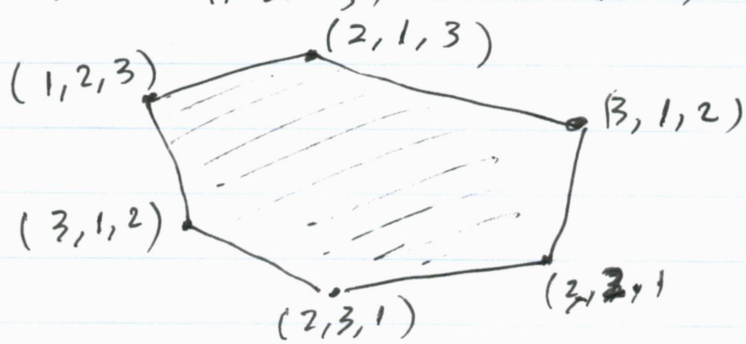
$$\mathcal{H}_\lambda := \{ A \text{ Hermitian w/ spectrum } \lambda \}$$

"isospectral set"

Consider $\Phi : \mathcal{H}_\lambda \longrightarrow \mathbb{R}^n \qquad \Phi(A) = \mathrm{diag}(A)$

<u>Thm</u>: image(Φ) = convex hull of permutations of λ

e.g. If $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3)$



Schur 1923 : $\subseteq$; Horn 1954 $\supseteq$

a proof in a book by Barvinok (A Course on Convexity) uses "Birkhoff - von Neumann thm" about {doubly stochastic matrices} = {convex hull of permutation matrices}

Goal: Use this theorem as excuse to talk about some objects / theorems in symplectic geometry.

Schur-Horn theorem follows from the following (classical) result in symplectic geometry:

thm [Atiyah; Guillemin-Sternberg]

$$U(1)^k = (S^1)^k = T \curvearrowright M$$ – compact, connected "symplectic manifold"

(group-action on M)

Suppose action admits a "moment map" $\Phi : M \to \cancel{\mathfrak{t}^*} \cong \mathbb{R}^n$

Then image(Φ) = convex hull of T-fixed points M^T

$M^T = \{m \in M : t \cdot m = m \ \forall t\}$

Next: • symplectic manifold
• moment map ~~(a bit of Lie theory?)~~

Consider first special case : 2×2 Hermitian matrices with spectrum $\lambda = (1,0)$.

$\mathcal{H}_\lambda = ?$

Every matrix in $\mathcal{H}_\lambda$ can be expressed as $A = P\begin{bmatrix}1&0\\0&0\end{bmatrix}P^*$ for some $P \in U(2)$.

So $\mathcal{H}_\lambda = \left\{ P\begin{bmatrix}1&0\\0&0\end{bmatrix}P^* : P\in U(2)\right\}$

<u>Fact</u>: Every matrix in $U(n)$ is a scalar multiple of one in $SU(n)$, and the scalar has norm 1.
ie. $P = \zeta P'$ where $\zeta \in S^1 = \{|z|=1 ; z\in\mathbb{C}\}$ and $P' \in SU(2)$.

So $\mathcal{H}_\lambda = \left\{ P\begin{bmatrix}1&0\\0&0\end{bmatrix}P^* : P\in SU(2)\right\}$
$\longrightarrow P = \begin{pmatrix} a & b \\ -\bar b & \bar a\end{pmatrix}$ $|a|^2+|b|^2=1$, $a,b\in\mathbb{C}$

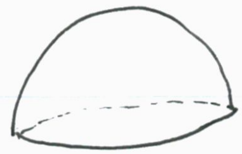
$$= \left\{ \begin{pmatrix} a\bar a & -ab \\ -\bar a\bar b & b\bar b \end{pmatrix} : \begin{array}{c}|a|^2+|b|^2=1\\ a,b\in\mathbb{C}\end{array}\right\}$$

Note this description "overcounts" (e.g. if $b=0$, there is an S^1-worth of descriptions of $\begin{pmatrix}1&0\\0&0\end{pmatrix}$.)

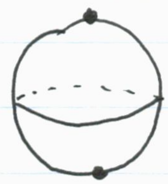
In fact, notice that $\begin{pmatrix} a\bar a & -ab \\ -\bar a\bar b & b\bar b \end{pmatrix} = \begin{pmatrix} \alpha\bar\alpha & -\alpha r \\ -\bar\alpha r & r^2\end{pmatrix}$

where $r = \sqrt{b\bar b}$, $\alpha = ab/r$; so we can take b to be real.

$|a|^2 + r^2 = 1$, $r \geq 0$

i.e. $\mathcal{H}_\lambda$ =  ← if $r=0$, these all represent $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, so they get identified together

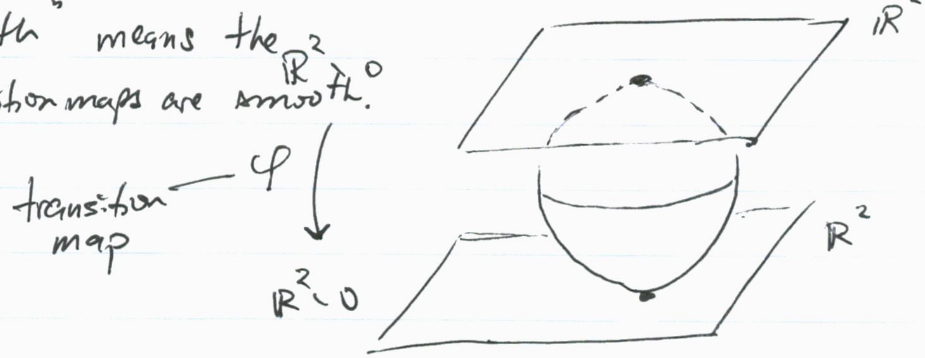
$\begin{bmatrix}0&0\\0&1\end{bmatrix}$

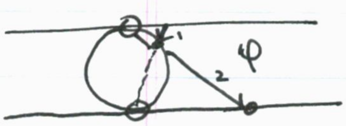
$\Rightarrow$ $\mathcal{H}_\lambda$ =  = S^2 (a manifold!)

$\begin{bmatrix}1&0\\0&0\end{bmatrix}$

Recall: • a manifold is a top. space locally homeo. to $\mathbb{R}^n$

• "smooth" means the transition maps are smooth. $\mathbb{R}^2 \setminus 0$





The T-action is easy to see:

$$\begin{tikzcd} U(n) \arrow[r] & \mathrm{Diff}(\mathcal{H}_\lambda) \\ \text{diagonals in } U(n) = T \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

(conjugation)

(conjugation by diagonal matrices in $U(n)$.)

Note: (n=2 case)

$$\begin{bmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{bmatrix} \begin{bmatrix} a\bar{a} & -ab \\ -\bar{a}b & b^2 \end{bmatrix} \begin{bmatrix} e^{-i\theta_2} & \\ & e^{-i\theta_2} \end{bmatrix} = \begin{bmatrix} a\bar{a} & -ab e^{i(\theta_1-\theta_2)} \\ & b^2 \end{bmatrix}$$

So action is rotation leaving N/S poles fixed.

Symplectic structure:
- on $\mathbb{R}^n$: a skew-symmetric bilinear non-deg. form.
- on a mfld M,

For each $p \in M$, a skew symmetric bilinear form $\omega(\ ,\ )_p$ on $T_pM \simeq \mathbb{R}^n$ that is non-degenerate, varying smoothly with p.

corresponding matrix is invertible

e.g. on $\mathbb{R}^2$: $\omega\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

or equivalently: $= [a \ b] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$

Fact: Under suitable choice of basis, every symplectic form has matrix rep. $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$\Rightarrow$ dim is even

a.k.a. "Hamiltonian vector fields"

Analogous to gradients, define "symplectic gradients" of functions $f: M \to \mathbb{R}$.

gradient: $\langle \nabla f, v \rangle = D_v f$; symp grad.: $\omega(X_f, v) = D_v f$

∇f perp to level curves

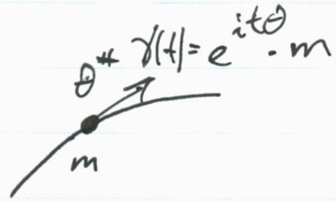
X_f tangent to level curves (set $v = X_f \Rightarrow D_v f = 0$)

Note: a smoothly varying symmetric non-deg. (PDS) bilinear form would give a pseudo-Riemannian metric. (Riemannian, metric, resp.)

Moment(um) maps

Given action $\underset{\overset{\shortparallel}{(S^1)^n}}{T} \longrightarrow \mathrm{Diff}_\omega(M)$ (preserving symp. structure)

Consider $\theta = (\theta_1, \dots, \theta_n)$ $\quad e^{i\theta} = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$



Let $\theta^\# = \left.\frac{d}{dt}\right|_{t=0} e^{it\theta} \cdot m$.

"generating vector field for action"

(Loosely speaking) the action admits a moment map if each $\theta^\#$ is a symplectic gradient (analogue of "conservative vector fields")

i.e. $\exists$ function f_θ with $\omega(\theta^\#, v) = D_v f_\theta$

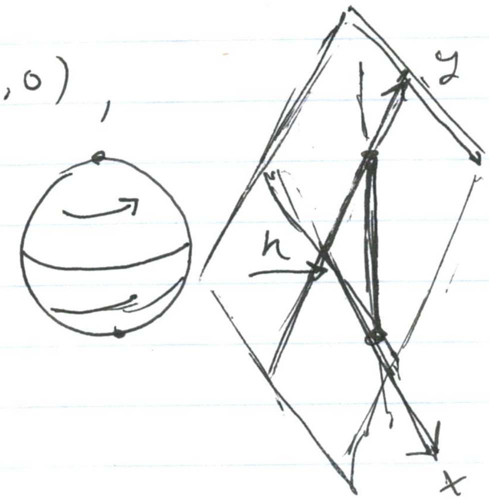
If so, moment map $\Phi : M \longrightarrow \mathbb{R}^n$ is $\Phi = (f_{e_1}, \dots, f_{e_n})$ ($e_1, \dots, e_n$ std basis vectors).

symp. potential means something else.

- kind of like a "symplectic primitive ~~(or potential)~~" of vector fields $\theta^\#$; <u>note</u>: these come from group actions, so there is a bit more to the story.

Returning to $\mathcal{H}_\lambda$, $\lambda = (1,0)$,

action is rotation,
moment map is essentially height



In general, it turns out moment map for

T-action on $\mathcal{H}_\lambda$ (by conj.) is $\Phi : \mathcal{H}_\lambda \longrightarrow \mathbb{R}^n$

$A \longmapsto \mathrm{diag}(A)$.

T-fixed points: $DAD^{-1} = A$ for all D diagonal
$\iff$ A diagonal.

$$\Rightarrow \mathcal{H}_\lambda^T = \{\text{permutations of } \lambda\}.$$

(So Atiyah / Guillemin-Sternberg Φ's theorem proves the Schur-Horn theorem.)