THE UNIVERSITY OF MANITOBA

December 13, 2005 — FINAL EXAMINATION

PAPER NO: 275 — TITLE PAGE

DEPARTMENT & COURSE NO: 136.150 — TIME: 2 HOURS

EXAMINATION: Introductory Calculus — EXAMINER: (Identified Below)

NAME: (PRINT) ______________________________

STUDENT NUMBER: ____________________

SIGNATURE: ______________________________
(I understand that cheating is a serious offense)

Please indicate your instructor and section by placing a check mark in the appropriate box below.

SECTION		TIME	INSTRUCTOR
☐ L01	M,W,F Tues.	10:30 - 11:20 10:00 - 10:50	Penner
☐ L02	M,W,F	9:30 - 10:20	Shivakumar
☐ L03	Tues, Thurs.	10:00 - 11:15	Kalajdzievski
☐ L04	M,W,F.	11:30 - 12:20	Korytowski
☐ L05	M,W,F.	12:30 - 1:20	Gumel
☐ L06	M,W,F.	3:30 - 4:20	Young
☐ L07	Tues. Even.	7:00 - 9:45	Sichler
☐ L91	Challenge for credit ☐ Dakota ☐ St. John's Ravenscourt ☐ Sisler		

INSTRUCTIONS TO STUDENTS:

This is a 2 hour exam. Please show your work ***clearly.***

No calculators, texts, notes or other aids are permitted.

This exam has a title page, 7 pages of questions and 2 blank pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staples.

The value of each question is indicated in the left-hand margin beside the statement of the question. The total value of all questions is 120.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

DO NOT WRITE IN THIS COLUMN

1.	/14
2.	/11
3.	/14
4.	/ 10
5.	/10
6.	/22
7.	/12
8.	/8
9.	/11
10.	/8
TOTAL	/120

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DEPARTMENT & COURSE NO: 136.150 | TIME: 2 HOURS

EXAMINATION: Introductory Calculus | EXAMINER: (Various)

Values

1. Compute $f'(x)$. Do not simplify your answer.

[4] a) $f(x)=\sin(e^{\ln x})+\ln(e^{2005})$

$= \sin(x) + 2005$

So $f'(x) = \cos(x)$

Recall: $e^{\ln x} = x$

$\ln(e^x) = x$

[4] b) $f(x)=\tan\left(\frac{x^2}{\cos x}+1\right)$

$$f'(x) = \sec^2\left(\frac{x^2}{\cos x}+1\right)\cdot\left(\frac{x^2}{\cos x}+1\right)'$$

$$f'(x) = \sec^2\left(\frac{x^2}{\cos x}+1\right)\cdot\left(\frac{2x\cos(x) - x^2(-\sin(x))}{(\cos(x))^2}\right)$$

[6] c) $f(x)=x^{2\cos(x^2)}$

$$f(x) = \left(e^{\ln x}\right)^{2\cos(x^2)} = e^{\ln(x)\cdot 2\cos(x^2)}$$

so $$f'(x) = e^{\ln(x)\cdot 2\cos(x^2)}\cdot\left(\underbrace{2\ln(x)}\,\underbrace{\cos(x^2)}\right)'$$

$$= e^{\ln(x)\cdot 2\cos(x^2)}\cdot\left(\frac{2}{x}\cos(x^2) + 2\ln(x)\left(-\sin(x^2)\cdot 2x\right)\right)$$

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Values

2.

[2] a) State when a function $f(x)$ is continuous at $x = a$.

A function is continuous at $x=a$ if $\lim\limits_{x\to a} f(x) = f(a)$.

[2] b) State when a function $f(x)$ is differentiable at $x = a$.

A function $f(x)$ is differentiable at $x=a$ if $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists, or equivalently $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists.

[7] c) Prove that if a function $f(x)$ is differentiable at $x = a$ then it is continuous at $x = a$.

Note that

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a} \cdot x-a.$$

So taking limits of both sides

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right] \text{ applying limit laws}$$

$$\Rightarrow \lim_{x\to a} f(x) - \lim_{x\to a} f(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} x-a$$

$$\lim_{x\to a} f(x) - f(a) = f'(a) \cdot 0 \quad \text{exists by assumption}$$

$$= 0$$

Therefore $\lim\limits_{x\to a} f(x) = f(a)$, and f is cts.

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Values

3.

[7] **a)** Find all points (a,b) on the curve $y = x^3 - x + 1$ where the tangent line is parallel to the line $y = 11x + 5$.

$y' = 3x^2 - 1$. For what x is $3x^2 - 1 = 11$?

$\Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$.

Therefore $y = (-2)^3 - (-2) + 1 = -8 + \overset{2}{\cancel{4}} + 1 = -3$

$y = (2)^3 - (2) + 1 = 8 - 1 = 7$. So $(2,-5)$ and $(2,7)$.

[7] **b)** Compute y' at the point $(1,1)$ if $3y^3x^2 - 3xy + 2x = 2$.

Implicit: $3(3y^2y'x^2 + \cancel{3}y^3 \cdot 2x) - 3(yy'x + y) + 2 = 0$

So with $x=1$, $y=1$ we get: $3(3y' + 2) - 3(y' + 1) + 2 = 0$

$\Rightarrow 6y' + 6 - 3 + 2 = 0$

$6y' = -5$, $y' = -5/6$

4.

[5] **(a)** Compute $f''(x)$ if $f(x) = 9\log_{10}\left(\frac{x}{3}\right)$.

Recall $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$, from $\log_a(x) = \frac{\ln(x)}{\ln(a)}$. therefore

$f'(x) = 9\frac{1}{\left(\frac{x}{3}\right)\ln(10)} \cdot \frac{1}{3} = \frac{9}{\ln(10)} \cdot \frac{1}{x}$, so $f''(x) = \frac{-9}{\ln(10)} \cdot \frac{1}{x^2}$

[5] **(b)** Suppose $f(x) = 2^{3x}$. First find $f'(x)$, $f''(x)$ and $f'''(x)$, and then use the pattern you see to compute $f^{(1000)}(x)$ (the 1000th derivative of $f(x)$). **DO NOT** simplify your answer.

Recall $\frac{d}{dx}(2^x) = 2^x \cancel{\log}(2)$ so $f'(x) = 2^{3x}\ln(2)\cdot 3$

Then $f''(x) = \ln(2)\cdot 3\left(2^{3x}\ln(2)\cdot 3\right) = (\ln(2)\cdot 3)^2\, 2^{3x}$

$f'''(x) = (\ln(2)\cdot 3)^2\, 2^{3x} \cdot \ln(2)\cdot 3 = (\ln(2)\cdot 3)^3\, 2^{3x}$ and following the pattern

$f^{(1000)}(x) = (\ln(2)\cdot 3)^{1000} \cdot 2^{3x}$

Values

[10] **5.** Find the absolute minimum and the absolute maximum of the function $f(x) = x^3 - 3x^2 - 9x + 2$ over the interval $[-2,2]$.

We get $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$.

So we get one crit pt $x = -1$ in the interval $[-2,2]$. We test:

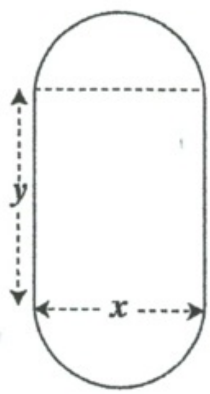
$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 2 = -8 - 12 + 18 + 2 = 0$

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = -1 - 3 + 9 + 2 = 7$

$f(2) = 2^3 - 3\cdot 4 - 9\cdot 2 + 2 = 8 - 12 - 18 + 2 = -20$

So $x = 2$ gives a min of $f(2) = -20$ and $x = -1$ gives a max of $f(-1) = 7$.

[12] **6.** A window is in the shape of a rectangle surmounted on both sides by semicircles as in the picture to the right. If the perimeter of the window is 6 m find the dimensions x and y (as shown in the figure) so that the greatest amount of light is admitted through the window.



circumference $= 2\pi r = \pi \cdot d = \pi x$.

Ie. maximize the area. The total area is

$A = xy + \pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi}{4}x^2$.

And $6 = 2y + \pi x \Rightarrow y = \frac{6 - \pi x}{2} = 3 - \frac{\pi}{2}x$. Thus

$A = x\left(3 - \frac{\pi}{2}x\right) + \pi\left(\frac{x^2}{4}\right) = 3x - \frac{\pi}{2}x^2 + \frac{\pi}{4}x^2 = 3x - \frac{\pi}{4}x^2$.

Note $x \geq 0$ and since $\pi x \leq 6$, $x \leq \frac{6}{\pi}$. So we test

$x = 0$, $x = \frac{6}{\pi}$, and the critical point coming from $A' = 3 - \frac{\pi}{2}x = 0$.

ie. $\frac{\pi}{2}x = 3 \Rightarrow x = \frac{6}{\pi}$

If $x = 0$ then $A = 3(0) - \frac{\pi}{4}\cdot(0)^2 = 0$, if $x = \frac{6}{\pi}$, then

$A = \frac{18}{\pi} - \frac{\pi}{4}\cdot\left(\frac{6}{\pi}\right)^2 = \frac{18}{\pi} - \frac{\pi}{4}\cdot\frac{36}{\pi^2} = \frac{18}{\pi} - \frac{9}{\pi} = \boxed{\frac{9}{\pi}}$

ie. A circular window of diameter

Values

[23] **7.** Suppose $f(x) = \ln(x^2+1)$. Then (no need to check) $f'(x) = \dfrac{2x}{1+x^2}$,

$$f''(x) = \frac{2(1-x^2)}{(1+x^2)^2}.$$

[18] a) Compile the following information about $f(x)$ and its graph. (Give answers only: Answer "none" if the function does not display a feature listed).

Need $x^2+1>0$, but this is always true

[1] Domain all of $\mathbb{R}$

[1] Symmetry (is $f(x)$ even, odd or neither ?) even

[1] Equation(s) of vertical asymptote(s) none.

[1] Equation(s) of horizontal asymptote(s) none.

[2] Coordinates of the critical point(s) of $f(x)$ $(0,0)$

[2] Interval(s) where $f(x)$ is increasing $(0,\infty)$ since $1+x^2>0\ \forall x$ and $2x>0$ here

[2] Interval(s) where $f(x)$ is decreasing $(-\infty,0)$ since $1+x^2>0\ \forall x$ and $2x<0$ here

[1] Coordinates of local maxima no maxima

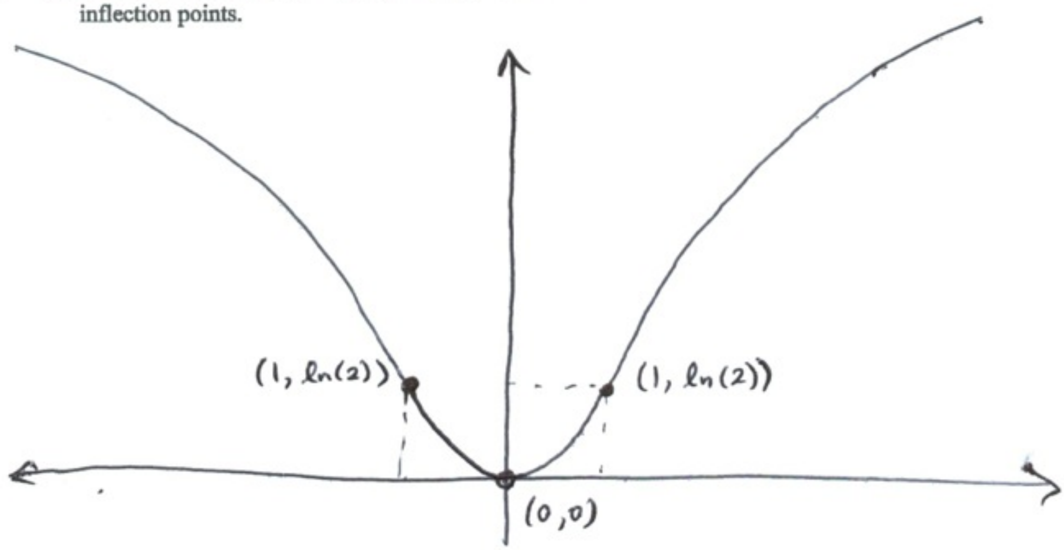
[2] Coordinates of local minima $(0,0)$

[2] Intervals where $f(x)$ is concave up $(-1,1)$

[2] Intervals where $f(x)$ is concave down $(-\infty,-1)\cup(1,\infty)$

[2] x-coordinates of inflection points $x=-1$ and $x=1$.

[4] b) Make a clear sketch of $y=f(x)$ labeling extreme and inflection points.



b)

$f(x) = \ln(x^2+1)$

$f(-x) = \ln((-x)^2+1) = \ln(x^2+1) = f(x)$

So $f(x)$ is even.

c) $\ln(x)$ has a vertical asymptote at $x=0$,
so $\ln(x^2+1)$ could have a vertical asymptote if $x^2+1=0$.
However $x^2+1=0$ is not possible $\Rightarrow$ no vert asymptote.

d) $\lim_{x\to\infty} \ln(x) = \infty$, so $\lim_{x\to\infty} \ln(x^2+1) = \infty$ and since it's even
$\lim_{x\to-\infty} \ln(x^2+1) = \infty$ so no horizontal asymptotes.

e) Critical pt if $f'(x)=0$ or undefined. Always defined, so only at $2x=0 \to x=0$. In this case $f(0) = \ln(0^2+1) = \ln(1) = 0$.
So crit pt. coords are $(0,0)$.

f) Concavity: The bottom $(1+x^2)^2$ is always positive. The top is $2(1-x^2) = 2(1-x)(1+x)$ so we get:

function	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$1-x$	+	+	−
$1+x$	−	+	+
$f''(x)$	−	+	−
$f(x)$	$\cap$ conc. down	$\cup$ conc up	$\cap$ conc down

Val:.es

[8] **8.** Find $g(x)$ if $g''(x)=12x+\frac{1}{x^2}$, $g'(1)=2$ and $g(1)=-2$.

Since $g''(x) = 12x + x^{-2}$, we get

$g'(x) = 12\left(\frac{x^2}{2}\right) + \frac{x^{-1}}{-1} + C = 6x^2 - \frac{1}{x} + C$. Then use $g'(1)=2$ and get

$2 = 6(1) - \frac{1}{1} + C \Rightarrow C = 2-6+1 = -3$. So

$g'(x) = 6x^2 - x^{-1} - 3$. Therefore

$g(x) = 6\left(\frac{x^3}{3}\right) - \ln(x) - 3x + D$, Then use $g(1) = -2$ to find D.

$= 2x^3 - \ln(x) - 3x + D$. (See opposite page)

9.

[5] (a) Evaluate $\int\left(x^5 - \sin x - \sqrt[3]{x}\right)dx$.

This means "find an antiderivative", which we should know how to do:
(general)

$$= \frac{x^6}{6} - (-\cos(x)) - \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= \frac{x^6}{6} + \cos(x) - \frac{3}{4}x^{4/3} + C.$$

[5] (b) Find $\frac{dF}{dx}$ if $F(x)=\int_0^{x^2-1}(t-3)\,dt$.

If $g(u) = \int_0^u (t-3)\,dt$ and $u(x) = x^2-1$, then

$F(x) = g(u(x))$. Therefore $F'(x) = \frac{dg}{du}\cdot\frac{du}{dx}$

FTC Part 1

$= (u-3)\cdot 2x$

ordinary derivative rules.

$= ((x^2-1)-3)\cdot 2x$

$= 2x^3 - 8x$.

Get:

$$-2 = 2(1)^3 - \ln(1) - 3(1) + D$$

$$= 2 - 0 - 3 + D$$

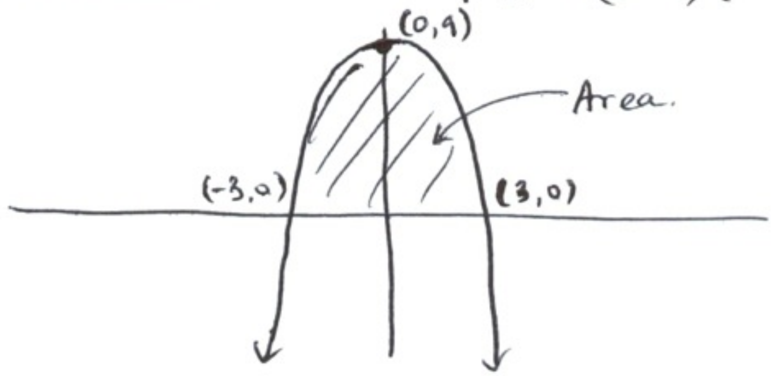
$$\Rightarrow D = -2 - 2 + 3 = -1.$$

So $g(x) = 2x^3 - \ln(x) - 3x - 1.$

Values

[8] **10.** Sketch and find the area of the region bounded (from below) by the x-axis and the graph of the function $y = 9 - x^2$.

$9 - x^2 = (3-x)(3+x)$



So the area is

$$A = \int_{-3}^{3} 9 - x^2 \, dx$$

$$= \left[9x - \frac{x^3}{3} \right]_{-3}^{3} = \left(\left(9(3) - \frac{3^3}{3} \right) - \left(9(-3) - \frac{(-3)^3}{3} \right) \right)$$

$$= (27 - 9) - (-27 + 9)$$

$$= 18 + 18 = 36.$$