MATH 1500 Lecture 25 March 10

End §4.1, start §4.2

Last day we saw:
A critical value of a function $f(x)$ is a number c such that $f'(c)=0$ or $f'(c)$ does not exist. If $f(c)$ is a max or a min of $f(x)$, then c is a critical value.

Example: Find the critical values of

$$f(x) = |x^2 - x|.$$

Solution: There is a trick to deal with absolute values! Since $|u| = \sqrt{u^2}$ (because $\sqrt{\ }$ is always pos)

We can rewrite $f(x) = \sqrt{(x^2-x)^2}$.

Then to find critical values, we calculate $f'(x)$:

$$f'(x) = \frac{1}{2}\left((x^2-x)^2\right)^{-1/2} \cdot (2(x^2-x)) \cdot (2x-1) \quad \left(\begin{matrix}\text{chain}\\ \text{rule}\end{matrix}\right)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{(x^2-x)^2}} \cdot \cancel{2}(x^2-x)(2x-1)$$

$$= \frac{(x^2-x)(2x-1)}{\sqrt{(x^2-x)^2}}$$

$$= \frac{(x^2-x)(2x-1)}{|x^2-x|}$$

So the derivative is undefined when $x^2 - x = 0$

i.e. $x(x-1) = 0$

so $x = 0$ or $x = 1$.

Also $f'(x) = 0$ when the top is zero, and $f'(x)$ is defined,

i.e. $(x^2-x)(2x-1) = 0$ and $x \neq 0$ or 1.

So $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus the critical values are $x = 0, \frac{1}{2}, 1$.

While we still need some tools for analyzing <u>local</u> maxes and mins, we can deal with <u>absolute</u> maxes and mins.

Procedure for finding the absolute max/min of $f(x)$ on an interval $[a,b]$:

① Find the critical values $c_1, c_2, \ldots, c_n$ in $[a,b]$.

② Plug the critical numbers $c_1, c_2, \ldots, c_n$ into f, also plug in the numbers a and b.

③ The largest of the values $f(c_1), f(c_2), \ldots, f(c_n)$, $f(a), f(b)$ from step ② is the absolute max, the smallest is the absolute min.

<u>Example</u>: Find the absolute maximum and minimum of $f(x) = \dfrac{\ln(x)}{x}$ on the interval $[1, e^2]$.

<u>Solution</u>: We first need critical values, so we take

derivatives. By the quotient rule:

$$f'(x) = \frac{x \cdot (\frac{1}{x}) - 1 \cdot \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}.$$

Now $f'(x)$ is undefined if $x=0$, but 0 is not in $[1, e^2]$, so we disregard it.

Next, $f'(x)=0 \iff \dfrac{1-\ln(x)}{x^2} = 0$

$$\Rightarrow 1 - \ln(x) = 0$$

$$\Rightarrow \ln(x) = 1, \text{ so } x = e.$$

So the only critical value in $[1, e^2]$ is $x = e$.

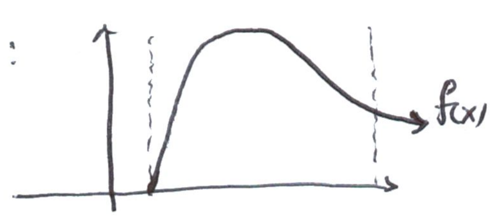
Next, we plug the numbers $[1, e^2]$ into f, and also the critical value $x=e$. We get:

$$f(1) = \frac{\ln(1)}{1} = \frac{0}{1} = 0$$

$$f(e) = \frac{\ln(e)}{e} = \frac{1}{e} \approx 0.3679$$

$$f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2} \approx 0.2707$$

The smallest of these is 0, so $x=1$ is where the absolute minimum value of $f(1)=0$ occurs. The largest of them is at $x=e$ where the absolute maximum of $f(e) = \frac{1}{e}$ occurs. Graph:



Example: Find the absolute max and min of $f(x)=\frac{x}{x^2+1}$ on $[0,2]$.

Solution: We need to calculate $f'(x)$ to find the critical values in $[0,2]$. We find:

$$f'(x)=\frac{(x^2+1)(x)'-x(x^2+1)'}{(x^2+1)^2}=\frac{x^2+1-x(2x)}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$$

Then $f'(x)$ is defined everywhere, so we only need to solve $f'(x)=0$. This gives

$$\frac{1-x^2}{(x^2+1)^2}=0 \iff 1-x^2=0$$

so $x^2=1$,

ie $x=\pm 1$.

The only one of ± 1 that is in the interval $[0,2]$ is $x=1$. So we need to check the values of $f(x)$ at $c=0,1,2$. We get:

$$f(0)=\frac{0}{0^2+1}=0,\quad f(1)=\frac{1}{1^2+1}=\frac{1}{2},\quad f(2)=\frac{2}{2^2+1}=\frac{2}{5}$$

So, of these values:

- $f(0)=0$ is the smallest so it is the absolute minimum.
- $f(1)=\frac{1}{2}$ is the largest, so it is the absolute maximum.

§4.2 The Mean Value Theorem.

This chapter is pure theory, meant to reinforce the theory supporting the calculations in the chapters that follow. Questions: 9-12, 19-21.

Mean Value Theorem: Suppose that $f(x)$ is a function, and

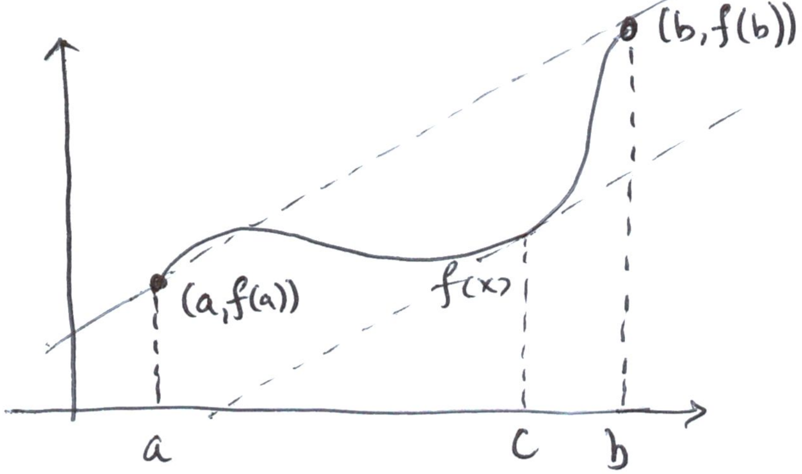
(i) $f(x)$ is continuous on $[a,b]$.

(ii) $f(x)$ is differentiable on (a,b)

Then there is a number c in (a,b) so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Explanation: Suppose we have such a function f:



What is the slope of the line connecting the endpoints of $f(x)$? It is

$$\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

The MVT says that there's a point c somewhere in (a,b) where the tangent line of f has this slope:

<u>Example</u>: Verify that $f(x) = x^3$ on $[0,2]$ satisfies the mean value theorem.

<u>Solution</u>: It's continuous and differentiable, so there should be a point c with

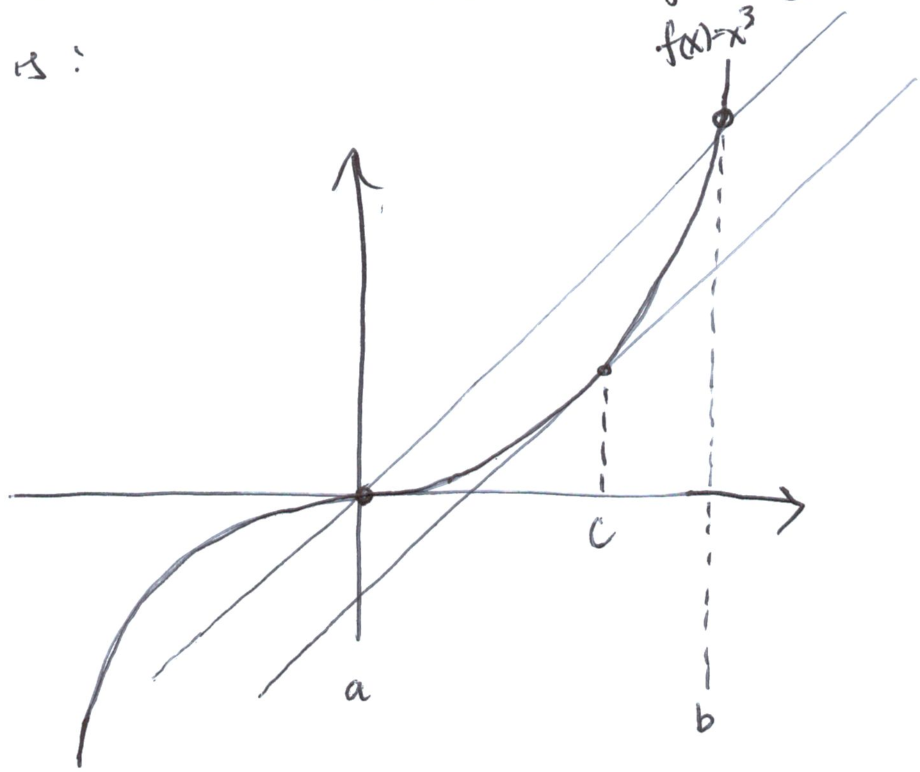
$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{2^3 - 0}{2-0} = \frac{8}{2} = 4.$$

Indeed, $f'(x) = 3x^2 = 4$

$$\Rightarrow x^2 = \frac{4}{3}, \text{ so } x = \pm \frac{2}{\sqrt{3}}.$$

Since $\frac{2}{\sqrt{3}}$ is in the interval $[0,2]$, we take the positive solution. Thus we have found $c = \frac{2}{\sqrt{3}}$.

The picture is:



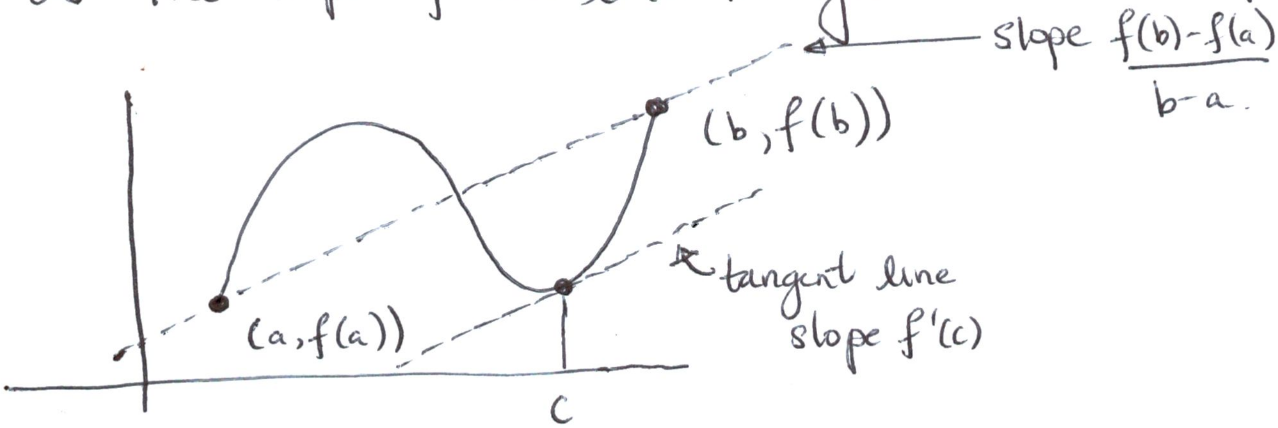
MATH 1500 March 12.

§ 4.2 Questions 9-12, 19-21.

The Mean Value Theorem:

Suppose that $f(x)$ is a function satisfying:
(i) $f(x)$ is continuous on $[a,b]$
(ii) $f(x)$ is differentiable on (a,b)
Then there's a number c so that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

I.e.: If $f(x)$ is some graph connecting the points $(a, f(a))$ and $(b, f(b))$, then somewhere along the curve $f(x)$ the slope of the tangent line is the same as the slope of the line through the start/end pts.



Example: $f(x) = x^3$ on $[0, 2]$ satisfies the hypotheses of the mean value theorem: it is continuous and differentiable everywhere. Therefore there should be a point c where the tangent line has slope

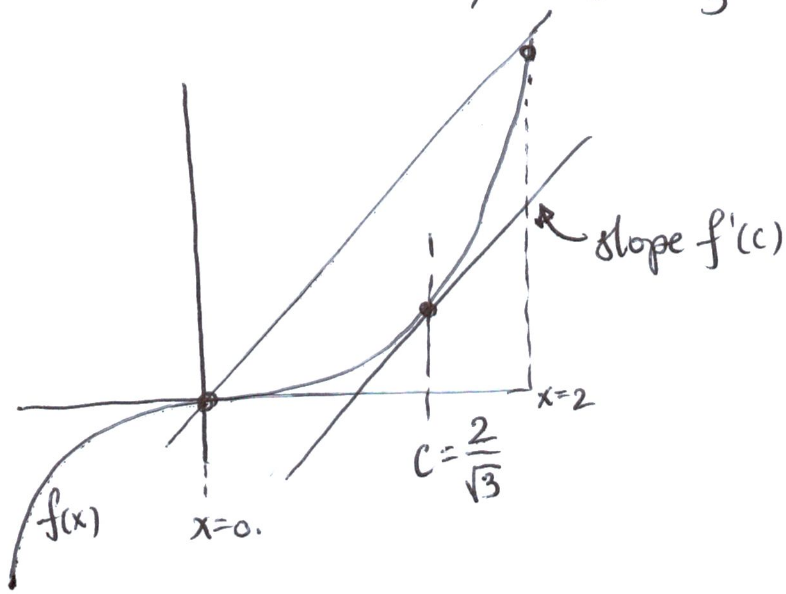
$$\frac{f(2)-f(0)}{2-0} = \frac{2^3-0^3}{2-0} = \frac{8}{2} = 4,$$

according to the Mean Value theorem. The exact value of c is: $f'(x) = 3x^2$, so

$$f'(c) = 4$$

$$\Rightarrow 3c^2 = 4, \text{ so } c^2 = \frac{4}{3} \Rightarrow c = \frac{2}{\sqrt{3}}.$$

Eg.



The application of the Mean Value theorem that concerns us is the following required proof:

$$f'(x)=0 \Rightarrow f(x) \text{ constant}$$

Proof: Pick any two numbers x_1 and x_2 in the interval (a,b), say x_1 is the smaller one so $x_1 < x_2$. Applying the Mean Value theorem to $f(x)$ on the interval (x_1, x_2), there exists a number c so that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

But remember $f'(x)=0$ everywhere, so

$$0 = f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \Rightarrow f(x_2)-f(x_1)=0$$

Therefore $0=f(x_2)-f(x_1)$, or $f(x_2)=f(x_1)$. But this means that any two numbers x_1, x_2 in (a,b) are sent to the same number by f, so $f(x)$ must be constant.

═══ END PROOF ═══

§ 4.3 Questions 1-29

What does $f'(x)$ tell us about the graph of $f(x)$?

Theorem: (i) If $f'(x)>0$ ~~for~~ on an interval (a,b), then $f(x)$ is increasing on (a,b).

(ii) If $f'(x)<0$ on an interval (a,b), then $f(x)$ is decreasing on (a,b).

Proof: We show that (i) is true.

A function f is called increasing if whenever $x<y$, $f(x)<f(y)$. So for our particular function $f(x)$; we need to show that if x_1, x_2 are points in (a,b) with $x_1<x_2$, then $f(x_1)<f(x_2)$.

Since we are told $f'(x)>0$ on (a,b), $f(x)$ is differentiable on the interval $[x_1, x_2] \subset (a,b)$.

So, by the Mean Value theorem there is a number c with $x_1 < c < x_2$ and

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}, \text{ or}$$

$$f(x_2)-f(x_1) = f'(c)(x_2-x_1).$$

But x_2-x_1 and $f'(c)$ are both positive, so the RHS is positive. Therefore

$$f(x_2)-f(x_1) > 0$$

$$\text{or } f(x_2) > f(x_1),$$

which means $f(x)$ is an increasing function.

<u>Example</u>: Where is the function

$$f(x) = x^4 - \frac{4}{3}x^3 - 12x^2 + 1$$

increasing? Where is it decreasing?

<u>Solution</u>: We find out where it is increasing/decreasing by examining $f'(x)$. We get

$$f'(x) = 4x^3 - \frac{4}{3}\cdot 3x^2 - 12\cdot 2x + 0$$

$$= 4x^3 - 4x^2 - 24x$$

$$= 4x(x^2 - x - 6) = 4x(x-3)(x+2).$$

So we need to know where this function $f'(x)$ is pos or negative. <u>Make a table</u>:

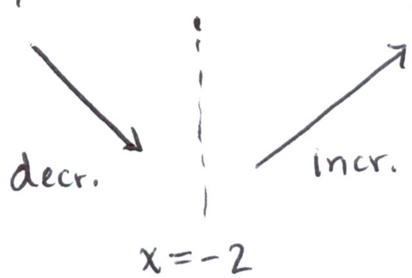
The numbers $-2, 0, 3$ are solutions to $f'(x) = 0$.
Use them to cut the real line into intervals
$(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$.

Then the table is:

function	$(-\infty, -2)$	$(-2, 0)$	$(0, 3)$	$(3, \infty)$
$4x$	$-$	$-$	$+$	$+$
$x - 3$	$-$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$	$+$
$f'(x) = 4x(x-3)(x+2)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$ (dec.)	$\nearrow$ (inc)	$\searrow$ (dec)	$\nearrow$ (inc).

So the function $f(x)$ is increasing on $(-2, 0)$ and $(3, \infty)$, and decreasing on $(-\infty, -2)$ and $(0, 3)$.

This is how we test for local maxes and mins!
For example, we see that $f(x)$ behaves like:



(decreasing on left, increasing on right)

near $x = -2$, so $x = -2$ is a <u>local minimum</u>.

by similar reasoning we see that $x=0$ is a local max and $x=3$ is a local min.

This method of finding local maxes and mins is called the first derivative test.

First derivative test:

Suppose c is a critical number of f.

(a) If $f'(x)$ is positive immediately to the left of c and negative immediately to the right, then $f(x)$ looks like $\nearrow \searrow$ so $f(c)$ is a local max.

(b) If $f'(x)$ is negative immediately to the left of c and positive immediately to the right, then $f(x)$ looks like $\searrow \nearrow$ and $f(c)$ is a local min.

(c) If $f'(x)$ doesn't change sign at c, then $f(c)$ is not a local max or local min.

MATH 1500 March 13.

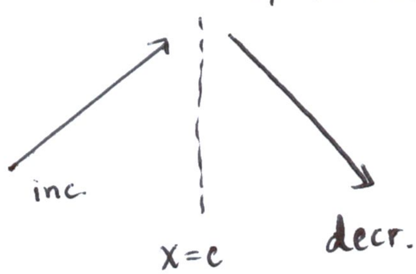
§4.3 continued.

Last day we saw that:

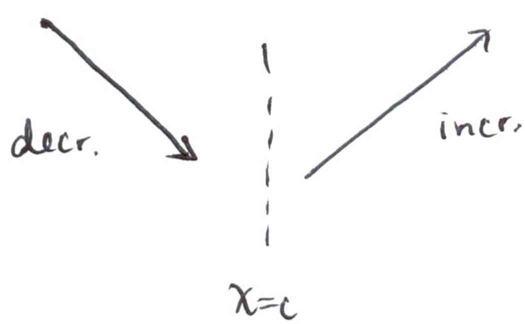
- If $f'(x) > 0$ on an interval (a,b) then $f(x)$ is increasing there
- If $f'(x) < 0$ on an interval (a,b) then $f(x)$ is decreasing there.

First derivative test:

Therefore, if $f'(x)$ changes from positive to negative at a critical value c, then $f(x)$ looks like:



So that $f(c)$ is a local max. Similarly if $f'(x)$ changes from negative to positive at a critical value c then $f(x)$ looks like



So $f(c)$ is a local minimum.

Example: If $f(x) = |x^2 - x|$, where are the local maxes and local mins?

Solution: We saw in a previous class that

$$f'(x) = \frac{x(x-1)(2x-1)}{|x^2-x|}$$ and the critical values are

$0, \frac{1}{2}, 1$. So we make a table (note that since $|x^2-x|$ is always positive we don't need it in the table:

function	$(-\infty, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 1)$	$(1, \infty)$
x	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$2x-1$	$-$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

min max min

So we know that $f(0) = 0$ is a local min, so is $f(1) = 0$. On the other hand

$$f\left(\tfrac{1}{2}\right) = \left|\left(\tfrac{1}{2}\right)^2 - \tfrac{1}{2}\right| = \left|\tfrac{1}{4} - \tfrac{1}{2}\right| = \tfrac{1}{4}$$

is a local max.

——— END EXAMPLE ———

Now we know what the first derivative $f'(x)$ says about $f(x)$. What about $f''(x)$?

<u>Definition</u>: A graph is <u>concave up</u> if it looks like

, ie. all of its tangent lines are below the graph:

This happens exactly when $f''(x) > 0$. If $f''(x) < 0$, then the graph is <u>concave down</u>:

The points where $f''(x) = 0$ (ie. $f(x)$ changes from concave down to convave up or vice versa) are called <u>inflection points</u>.

<u>Example</u>: Where is the function $f(x) = x^4 - 4x^3$ increasing, decreasing, concave up and concave down? What are the max/min and the inflection points?

<u>Solution</u>: The first derivative of f is

$$f'(x) = 4x^3 - 12x^2$$
$$= 4x^2(x-3)$$

So the critical values are $x=0$ and $x=3$.

So we get a table:

function	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$4x^2$	+	+	+
$x-3$	−	−	+
$f'(x)$	−	−	+
$f(x)$	$\searrow$ decr.	$\searrow$ decr	$\nearrow$ incr.

So $f(x)$ has a minimum at $f(3) = 3^4 - 4 \cdot 3^3 = -27$, is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

Then $f''(x) = 12x^2 - 24x$
$= 12x(x-2)$

So $f''(x) = 0$ gives $x = 0$ and $x = 2$. Then we get a table:

function	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$12x$	−	+	+
$(x-2)$	−	−	+
$f''(x)$	+	−	+
$f(x)$	$\cup$ conc. up.	$\cap$ conc. down	$\cup$ conc. up.

The inflection points are at $x = 0$ and $x = 2$, they are $f(0) = 0$ and $f(2) = 2^4 - 4(2^3) = -16$. The graph is concave down on $(0, 2)$, and concave up on $(-\infty, 0)$ and $(2, \infty)$.

Note we can also see from the second derivative that $f(3)$ is a local min, because $f(x)$ is concave up there.

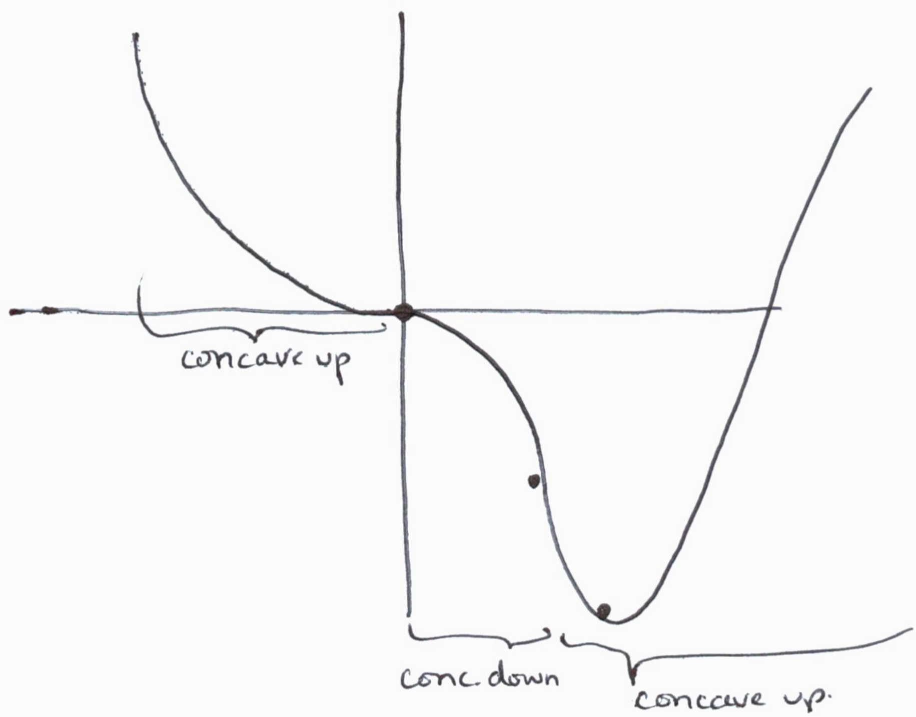
<u>Second derivative test</u>: Suppose f'' is cts near c.

(i) If $f'(c)=0$ and $f''(c)>0$, then $f(x)$ has a local minimum at c ($f(x)$ looks like )

(ii) If $f'(c)=0$ and $f''(c)<0$, then $f(x)$ has a local maximum at c ($f(x)$ looks like )

So we can sketch $f(x)$ by first:

- Plot the maxes/mins (min @ $(3,-27)$)
- Add the inflection points $(0,0)$ and $(2,-16)$.



Note that this curve is still somewhat "sketchy".
We will spend 2 more classes refining our sketching techniques.

Also: Notice that the second derivative test can fail!
If $f''(c)=0$, as happened in the previous example with $c=0$, we learn no information about whether or not c is a local max or local min.