MATH 1500 Calculus March 3

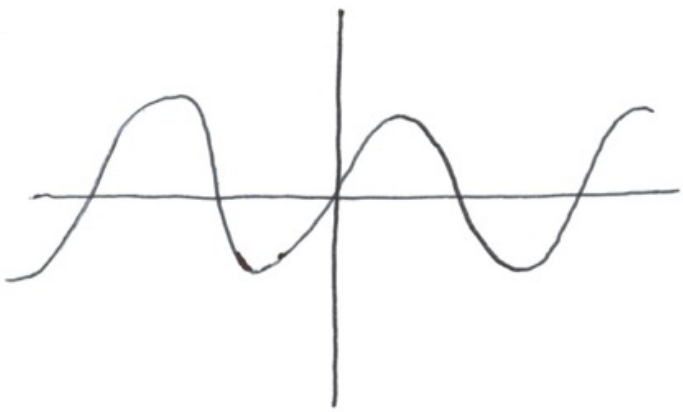
Lecture 22 §1.6-3.6

Last day we saw inverse functions and logarithms. Today, inverse trig functions. We begin with $\sin(x)$.

Recall that last day we saw:

The graph of $f^{-1}(x)$ is the same as $f(x)$, reflected in the line $y=x$.

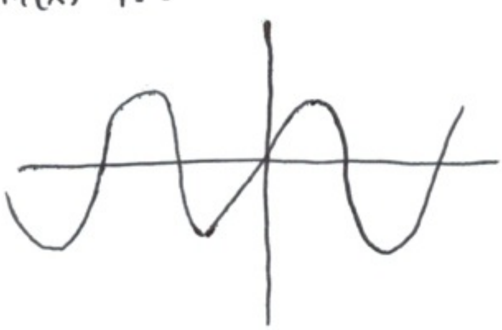
Since $\sin(x)$ looks like:



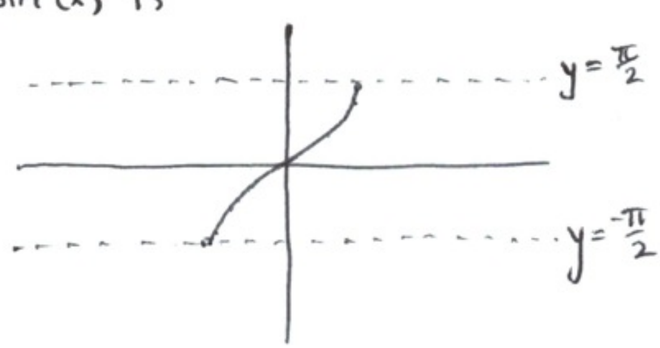
we would hope for $\sin^{-1}(x)$ to have graph:

However, this graph is not a function: for example it does not pass the vertical line test. So the best we can do is to take only a portion of the graph:

$\sin(x)$ is:



and $\sin^{-1}(x)$ is



In other words,

$$\sin^{-1}(x) = y \iff \sin y = x \text{ and } \frac{-\pi}{2} \le y \le \frac{\pi}{2}$$

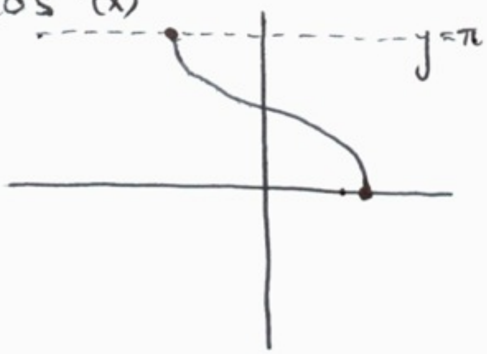
so that

$$\sin^{-1}(\sin(x)) = x \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

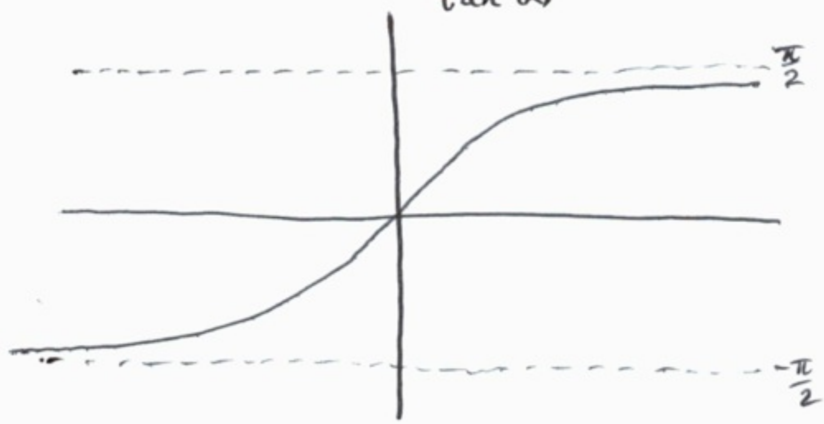
$$\text{and } \sin(\sin^{-1}(x)) = x \text{ for } -1 \le x \le 1.$$

The other trig functions $\cos^{-1}(x)$, $\tan^{-1}(x)$ are defined in a similar way, and their graphs are:

$\cos^{-1}(x)$



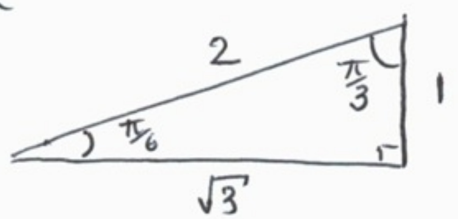
$\tan^{-1}(x)$



Example: In general, $\cos^{-1}(x)$, $\sin^{-1}(x)$, $\tan^{-1}(x)$ are numbers whose exact value we cannot calculate. In some cases, we know the answer based on triangles that we remember:

What is the exact value of $\tan^{-1}(\sqrt{3})$?

Recall

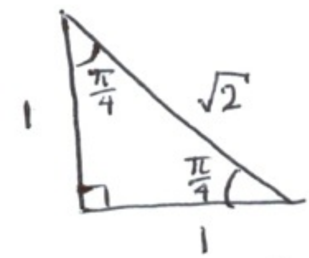


, so what angle θ gives $\tan(\theta) = \sqrt{3}$?

We see $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, so taking $\tan^{-1}$ of both sides

$$\frac{\pi}{3} = \tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right) = \tan^{-1}(\sqrt{3})$$

or the exact value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ comes from



and therefore $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, and $\cos^{-1}$ of both sides gives $\underbrace{\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)}_{= \frac{\pi}{4}} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Now we have a new collection of functions, log, ln and inverse trig, so we jump to § 3.6 to learn how to differentiate them:

The derivative of a^x is: $\frac{d}{dx}(a^x) = a^x \ln(a)$

On the other hand, $\log_a(x)$ is a number y satisfying

$$a^y = x$$

so implicit differentiation gives

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x)$$

$$\begin{pmatrix}\text{chain}\\ \text{rule}\end{pmatrix} \Rightarrow \quad a^y \ln(a) \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{a^y \ln(a)}, \quad \text{where } y = \log_a(x)$$

so pluggingin $y = \log_a(x)$ gives

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{a^{\log_a(x)}\ln(a)} = \frac{1}{x\ln(a)}.$$

and in particular if $a=e$ then

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x\ln(e)} = \frac{1}{x}.$$

<u>Example</u>: What is the slope of the line tangent to $y=\ln(2x^2+1)$ at $x=0$? what is its equation?

<u>Solution</u>: Using $\frac{d}{dx}\ln(x) = \frac{1}{x}$ and the chain rule,

we get $\frac{d}{dx}(\ln(2x^2+1)) = \frac{1}{2x^2+1}\cdot 4x = \frac{4x}{2x^2+1}$

so the slope of the tangent line at $x=0$ is

$$\frac{0}{2(0)^2+1} = 0.$$

The tangent line at $x=0$ passes through the point $x=0$, $y=\ln(2(0)^2+1) = \ln(1) = 0$.

So $y=mx+b$ with $m=0$, and we get

$$y=b$$

with $b=0$ so that the line goes through $(0,0)$.

So equation is $y=0$.

Example: Differentiate $\log_{10}(\cos(x^2)+5)$.

Solution: We use $\frac{d}{dx}(\log_{10}(x)) = \frac{1}{x\ln(10)}$

and the chain rule. So we get; with:

$f(u) = \log_{10}(u)$, $u(v) = \cos(v)+5$, $v(x) = x^2$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}, \text{ and}$$

$$\frac{df}{du} = \frac{1}{u\ln(10)}, \quad \frac{du}{dv} = -\sin(v), \quad \frac{dv}{dx} = 2x$$

So

$$\frac{df}{dx} = \frac{1}{u\ln(10)} \cdot (-\sin(v)) \cdot 2x,$$ changing everything to x gives

$$= \frac{1}{(\cos(x^2)+5)\ln(10)} \cdot (-\sin(x^2)) \cdot 2x$$

$$= \frac{-2x\sin(x^2)}{(\cos(x^2)+5)\ln(10)}.$$

We can also differentiate complicated expressions by taking ln of both sides first, this trick is called logarithmic differentiation.

Example: Differentiate $y = x^{\sqrt{x}}$.

Solution: Take ln of both sides. Then

$$\ln(y) = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\,\ln(x)$$

and now differentiate:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\sqrt{x}\,\ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{d}{dx}(\sqrt{x})\right]\ln(x) + \sqrt{x}\,\frac{d}{dx}(\ln(x))$$

$$= \frac{1}{2}x^{-1/2}\ln(x) + \sqrt{x}\cdot\frac{1}{x}$$

so

$$\frac{dy}{dx} = y\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right)$$

$$= x^{\sqrt{x}}\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right).$$

MATH 1500 March 5

Lecture 23

§3.6 Questions 2-22, 39-50.

Last day we saw that the derivatives of log functions are

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

and when $a=e$, this reduces to $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, since $\ln(e)=1$.

We also saw a glimpse of a trick called logarithmic differentiation, which we saw last day.

Example: Calculate y' if $y=(x^2+3x+1)^{18}(x^4-3)^6$.

Solution: We could use the product rule, or we could take ln of both sides and do logarithmic differentiation.

Product rule:

$$y' = \left((x^2+3x+1)^{18}\right)'(x^4-3)^6 + (x^2+3x+1)^{18}\left((x^4-3)^6\right)'$$

$$= 18(x^2+3x+1)^{17}\cdot(2x+3)\cdot(x^4-3)^6 + (x^2+3x+1)^{18}\cdot(4x^3)\cdot(6(x^4-3)^5)$$

$$= 18(x^2+3x+1)^{17}(x^4-3)^6(2x+3) + 24(x^2+3x+1)^{18}(x^4-3)^5\cdot x^3$$

Logarithmic differentiation: Take ln of both sides.

$$\ln(y) = \ln\left((x^2+3x+1)^{18}(x^4-3)^6\right)$$

$$= 18\ln(x^2+3x+1) + 6\ln(x^4-3)$$

Then doing $\frac{d}{dx}$ of both sides gives.

$$\frac{1}{y}\frac{dy}{dx} = 18\frac{d}{dx}\left(\ln(x^2+3x+1)\right) + 6\frac{d}{dx}\left(\ln(x^4-3)\right)$$

$$\Rightarrow \frac{dy}{dx}\cdot\frac{1}{y} = \frac{18(2x+3)}{(x^2+3x+1)} + \frac{6(4x^3)}{(x^4-3)}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{18(2x+3)}{(x^2+3x+1)} + \frac{24x^3}{x^4-3}\right)$$

$$= (x^2+3x+1)^{18}(x^4-3)^6\left(\frac{18(2x+3)}{(x^2+3x+1)} + \frac{24x^3}{x^4-3}\right)$$

$$= 18(x^2+3x+1)^{17}(x^4-3)^6(2x+3) + 24x^3(x^2+3x+1)^{18}(x^4-3)^5.$$

Example: Take the derivative of $y = \sqrt{\frac{x^2-1}{x^4+3}}$ using logarithmic differentiation.

Solution: Take ln of both sides. We get:

$$\ln(y) = \ln\left(\sqrt{\frac{x^2-1}{x^4+3}}\right) = \frac{1}{2}\ln(x^2-1) - \frac{1}{2}\ln(x^4+3)$$

So taking derivatives, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x^2-1}\cdot 2x - \frac{1}{2}\frac{1}{x^4+3}\cdot 4x^3$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{2}\left(\frac{2x}{x^2-1} - \frac{4x^3}{x^4+3}\right)\right)$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{x^2-1}{x^4+3}}\left(\frac{x}{x^2-1} - \frac{2x^3}{x^4+3}\right).$$

Example: If $f(x) = \ln|x|$, what is $f'(x)$?

Solution: Normally, absolute value gives problems when taking derivatives, since $|x|$ is not differentiable at $x=0$. Here we get two cases:

$$f(x) = \begin{cases} \ln(x) & \text{if } x>0 \\ \ln(-x) & \text{if } x<0. \end{cases}$$

So then if $x>0$, we get $f'(x) = (\ln(x))' = \frac{1}{x}$.

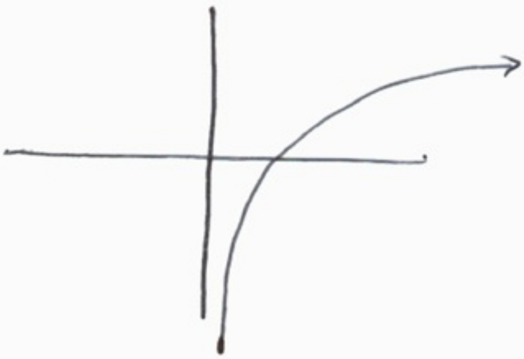
If $x<0$, we get $f'(x) = (\ln(-x))' = \underbrace{\frac{1}{-x} \cdot (-1)}_{\text{chain rule}} = \frac{1}{x}$.

So we get $f'(x) = \frac{1}{x}$. (there's still a problem at $x=0$)

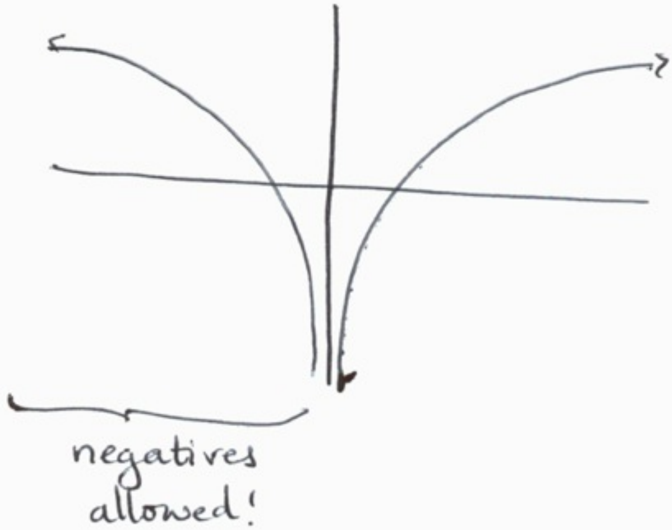
The point is that starting with $|x|$, upon differentiating we get a much simpler formula: If $f(x) = \ln|x|$, then $f'(x) = \frac{1}{x}$.

This last example means that by taking absolute value of x, we can plug negative numbers into $\ln|x|$ and differentiate, if we have to. "Normally" this is not possible, because working with $\ln(x)$ we can't plug in $(-x)$.

Graph of $\ln(x)$



Graph of $\ln|x|$



Example: Calculate the derivative of $f(x) = x^n$, using logarithmic differentiation.

Solution: We write $y = x^n$, then do $\ln|\ |$ of both sides. We use absolute value here because we want to allow for $x^n < 0$, e.g x^3 when $x < 0$.

Then

$$y = x^n \text{ gives } \ln|y| = \ln|x^n|$$

$$\Rightarrow \ln|y| = \ln|x|^n$$

$$= n \ln|x|.$$

So now, taking derivatives we get:

$$\frac{d}{dx} \ln|y| = \frac{d}{dx}(n \ln|x|)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = n \cdot \frac{1}{x}. \quad \text{So}$$

$$\frac{dy}{dx} = y \cdot \frac{n}{x}, \text{ but } y = x^n \text{ so}$$

$$\frac{dy}{dx} = x^n \cdot \frac{n}{x} = n x^{n-1}.$$

This works even if n is not integral!

<u>Recall</u>: We gave two definitions of the number e. The most significant was that we use e as the base of an exponential function $f(x)$ satisfying $f'(x) = f(x)$.

We can also show $e = \lim_{x \to 0} (1+x)^{1/x}$ using

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Substituting $n = \frac{1}{x}$, in which case $n \to \infty$ as $x \to 0$, we also get

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

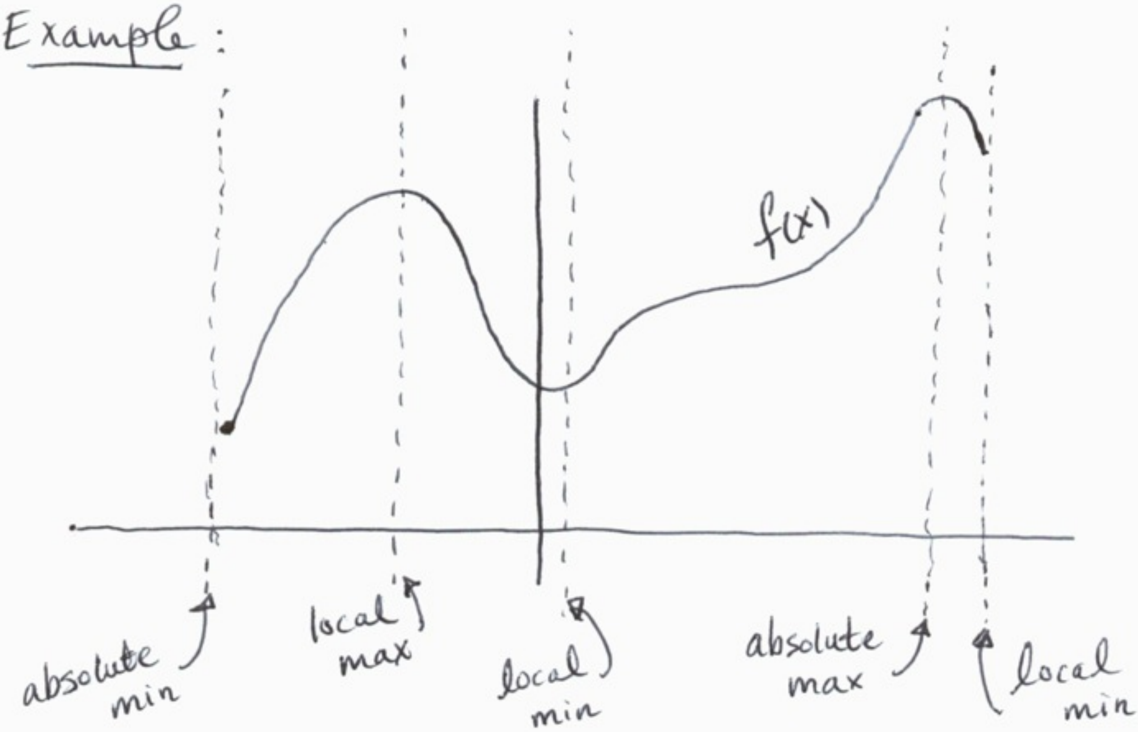
MATH 1500 Friday March 7.

§4.1 Questions 1–44

Today we calculate max and min values of a function, but first some terminology:

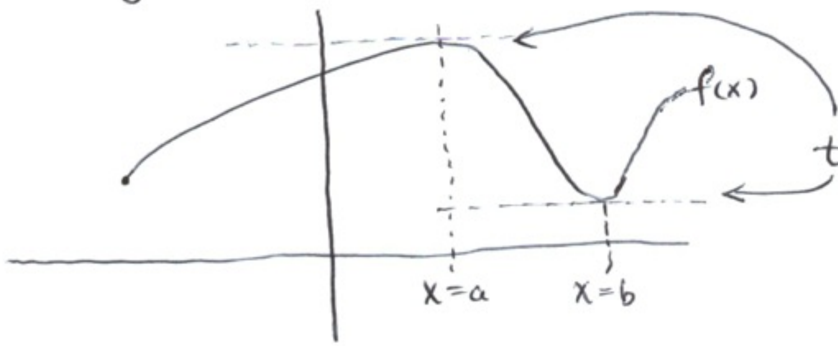
- If $f(x)$ is a function and $f(c)$ is the biggest (respectively smallest) number you can get by plugging in values from the domain of $f(x)$, then $f(c)$ is called an absolute maximum (respectively minimum).
- If $f(c)$ is the biggest (respectively smallest) number you get by plugging in numbers near c, then $f(c)$ is called a local maximum (respectively local minimum).

Example:



The maximum and minimum values of f are called extreme values of f.

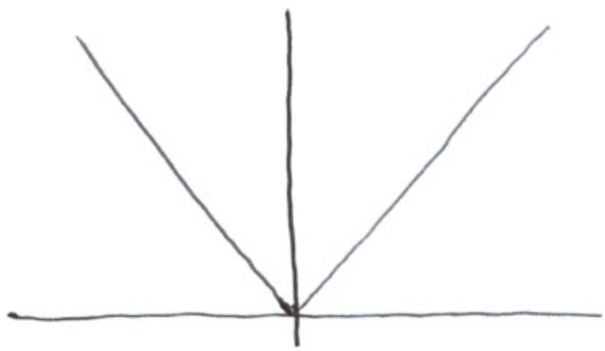
The way we detecte maxima and minima will be to use differentiation. Observe that at a max or min, the tangent line is flat:



tangent lines flat, so
$f'(a) = f'(b) = 0$

therefore we try to solve $f'(x)=0$ to find max/min.
We could also have:

$f(x) = |x|$

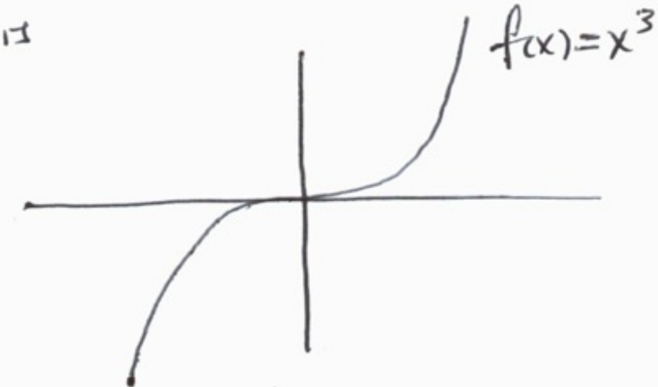


which has a minimum at a point where $f'(x)$ does not exist $(x=0)$. So we observe:

<u>Fact</u>: If $f(x)$ has a local maximum or minimum at $x=c$ then either $f'(c)=0$ or $f'(c)$ does not exist.
If $f'(c)=0$ or $f'(c)$ does not exist, then c is called a <u>critical point</u> of $f(x)$. So to find max and mins we find critical points!

Note: Just because $f'(c)$ doesn't exist or $f'(c)=0$, it doesn't mean we have a max or min there!

Example: The function $f(x)=x^3$ has derivative $f'(x)=3x^2$, so $f'(0)=3\cdot 0^2=0$ and $x=0$ is a critical point. However the graph is



which has no max or min at zero.

Example: Find the critical values of $f(x)=\dfrac{x^2}{x^2-1}$.

Solution: The derivative is

$$f'(x)=\frac{2x(x^2-1)-x^2(2x)}{(x^2-1)^2}=\frac{-2x}{(x^2-1)^2}=\frac{-2x}{((x+1)(x-1))^2}.$$

We need to find all values c s.t. $f'(c)=0$ or $f'(c)$ does not exist.

- Because $x^2-1=0$ causes division by zero, we get $(x+1)(x-1)=0 \Rightarrow x=\pm 1$ are values where f' doesn't exist.

- $\dfrac{-2x}{(x^2-1)^2}=0 \Rightarrow -2x=0 \Rightarrow x=0$. gives $f'=0$.

So the critical values of f are $x = \pm 1$, $x = 0$.

<u>Example</u>: Find the critical points of $f(x) = |x^2 - x|$.

<u>Solution</u>: We need to know where $x^2 - x > 0$ and where $x^2 - x < 0$ in order to keep track of the absolute value.

Since $x^2 - x = (x-1)x$, we can analyze the sign of $x^2 - x$ with a table: (noting only $0, 1$ are interesting).

function	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
x	$-$	$+$	$+$
$x - 1$	$-$	$-$	$+$
$x^2 - x$	$+$	$-$	$+$

Therefore $f(x) = |x^2 - x|$ can be described as a piecewise function

$$f(x) = \begin{cases} x^2 - x & \text{if } x \le 0 \\ -(x^2 - x) & \text{if } 0 \le x \le 1 \\ x^2 - x & \text{if } x \ge 1. \end{cases}$$

So then the derivative of each piece gives us

$$f'(x) = \begin{cases} 2x - 1 & \text{if } x < 0 \\ -2x + 1 & \text{if } 0 < x < 1 \\ 2x - 1 & \text{if } x > 1. \end{cases}$$

Now we test for two kinds of points:

- where does $f'(c)$ not exist?
- where is $f'(c)=0$?

Then: Testing $f'(c)=0$ we ask 3 cases:

① What $x<0$ makes $2x-1=0$?

Ans. no such x, so there is no $c<0$ with $f'(c)=0$.

② What x with $0<x<1$ makes $-2x+1=0$?

Ans: Solving, we get $-2x=-1$
$\Rightarrow x=\frac{1}{2}$.

$$\boxed{\text{So } f'(\tfrac{1}{2}) = -2(\tfrac{1}{2})+1 = 0.}$$

③ What $x>1$ makes $2x-1=0$?

Ans: no such x, so there is no $c>1$ with $f'(c)=0$.

Testing $f'(c)$ does not exist, there are two candidates:
$c=0$ and $c=1$.

At $c=0$ we see that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2x-1 = -1$$

and

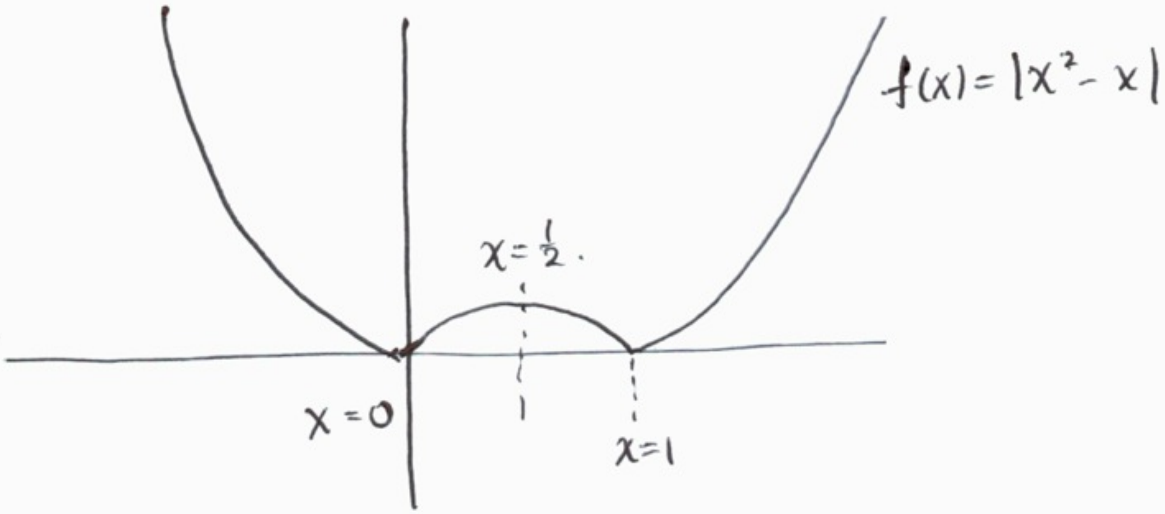
$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} -2x+1 = 1$$

So $f'(0)$ doesn't exist. Similarly $f'(1)$ doesn't exist.

Overall, the critical points are

$x=0, \frac{1}{2}, 1$.

The graph of $f(x) = |x^2 - x|$ is:



So indeed, we have found all places where max/min occur.

<u>Example</u>: What are the critical values of $f(x) = \tan(x)$?

<u>Solution</u>: We find $f'(x) = \sec^2(x)$.

So we need to know where $(\sec(x))^2 = \dfrac{1}{(\cos(x))^2}$ is zero and where it is defined.

First, it's obviously never zero since $\dfrac{1}{(\cos(x))^2} = 0$

$\iff 0 = 1$, which never happens.

It's not defined when $\cos(x) = 0$, which is at
$x = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc. or $x = \frac{\pi}{2} + k\pi$.

So the critical values of $\tan(x)$ are $\frac{\pi}{2} + k\pi$.