MATH 1500 Feb 24

Lecture 19 Midterm Review.

Winter 2005 midterm. Find the limits.

① a) $\lim_{x\to\infty} \dfrac{(2x^3+2)(x^2+3x)}{4x^5+5}$

$= \lim_{x\to\infty} \dfrac{2x^5+6x^4+2x^2+6x}{4x^5+5}$ multiply top and bottom by $\frac{1}{x^5}$.

$= \lim_{x\to\infty} \dfrac{2+\cancel{\frac{6}{x}}+\cancel{\frac{2}{x^3}}+\cancel{\frac{6}{x^4}}}{4+\cancel{\frac{5}{x^5}}}$

$= \dfrac{2}{4} = \dfrac{1}{2}$.

(b) $\lim_{x\to-\infty} \dfrac{\sqrt{4x^2+7x+2}}{3x+5}$. Multiply top and bottom by $\frac{1}{x}$.

$= \lim_{x\to-\infty} \dfrac{\frac{1}{x}\sqrt{4x^2+7x+2}}{\frac{1}{x}(3x+5)}$

To bring $\frac{1}{x}$ inside the square root, we replace $\frac{1}{x}$ with $\frac{1}{x} = -\sqrt{\frac{1}{x^2}}$, with a negative sign since $x \to \underline{\underline{-}}\infty$

$= \lim_{x\to-\infty} \dfrac{-\sqrt{4+\cancel{\frac{7}{x}}+\cancel{\frac{2}{x^2}}}}{3+\cancel{\frac{5}{x}}}$

$= \dfrac{-\sqrt{4}}{3} = \dfrac{-2}{3}$.

(c) $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$, Here setting $x=0$ gives $\frac{0}{0}$ so we must do some algebra.

$$= \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \cdot \left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$$

$$= \lim_{x\to 0} \frac{(x+4)-2^2}{x(\sqrt{x+4}+2}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2}$$

Now setting $x=0$ does not give $\frac{0}{0}$, instead we get

$$= \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}.$$

Question 2.

Find y'. Do not simplify.

(a) $y = x^{3/2} + x^{2/3} + x^{1/3}$

$$y' = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3}$$

(b) $y = \dfrac{1+\cos(x)}{1+(\sin(x))}. = (1+\cos(x))(1+\sin(x))^{-1}.$

By the product rule,

$$y' = (1+\cos(x))'(1+\sin(x))^{-1} + (1+\cos(x))((1+\sin(x))^{-1})'$$

$$= (-\sin(x))(1+\sin(x))^{-1} + (1+\cos(x))(-1)(1+\sin(x))^{-2}\cdot\cos(x)$$

$$= \frac{-\sin(x)}{1+\sin(x)} - \frac{\cos(x)(1+\cos(x))}{(1+\sin(x))^2}.$$

or $$\frac{-\sin(x)(1+\sin(x)) - \cos(x)(1+\cos(x))}{(1+\sin(x))^2}.$$

(c) $y = \sin(e^{x^2})$.

Then if $y = f(u) = \sin(u)$, $u(v) = e^v$, $v(x) = x^2$, then the chain rule gives

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}, \text{ so}$$

$$\frac{df}{du} = \cos(u), \quad \frac{du}{dv} = e^v, \quad \frac{dv}{dx} = 2x, \text{ so}$$

$$\frac{dy}{dx} = \cos(u) \cdot e^v \cdot 2x, \text{ or } = \cos(e^{x^2}) \cdot e^{x^2} \cdot 2x.$$

Question 3.

Consider $f(x) = \begin{cases} -3x & \text{if } x < -1 \\ -3 & \text{if } x = -1 \\ x^2+2 & \text{if } x > -1 \end{cases}$

(a) Does $\lim_{x \to -1}$ exist, if so what is its value?

Solution: We test left and right:

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -3x = (-3)(-1) = 3.$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2+2 = (-1)^2+2 = 3.$$

So yes, $\lim_{x \to -1} f(x)$ exists since the left and right are equal.

In fact, $\lim_{x \to -1} f(x) = 3$.

(b) Is $f(x)$ continuous at $x=-1$?

Solution: No. If it were continuous then $\lim_{x\to -1} f(x)$ would be equal to $f(-1)$, which is -3.

Question 4.

Prove the theorem:

If $f'(x)$ exists then $(cf(x))' = c\,(f'(x))$.

Solution:

By definition,

$$(cf(x))' = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h\to 0} \frac{c\,(f(x+h) - f(x))}{h}$$

$$= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{by limit laws.}$$

$$= cf'(x).$$

Question 5

Find the slope of the line tangent to the curve given by $x^2y^2 - x^3y^3 = 12$ at the point $(2,-1)$

Solution. We need y', we find it by implicit differentiation.

$$(x^2y^2 - x^3y^3)' = (12)'$$

Each of $(x^2y^2)'$ and $(x^3y^3)'$ is a product rule.

$$(x^2y^2)' = (x^2)'y^2 + x^2(y^2)' = 2xy^2 + x^2 2yy'$$

$$(x^3y^3)' = (x^3)'y^3 + x^3(y^3)' = 3x^2y^3 + x^3 3y^2y'$$

So we get

$$2xy^2 + x^2 2yy' - 3x^2y^3 - x^3 3y^2y' = 0.$$

or $x^2 2yy' - x^3 3y^2 y' = 3x^2y^3 - 2xy^2$

$$y' = \frac{3x^2y^3 - 2xy^2}{x^2 2y - x^3 3y^2}$$

Now set $x = 2,\ y = -1$.

$$y' = \frac{3(2)^2(-1)^3 - 2(2)(-1)^2}{(2)^2 2(-1) - (2)^3 3(-1)^2} = \frac{(-12-4)}{-8-24} = \frac{-16}{-32} = \frac{1}{2}$$

Question 6. Let $f(x) = \sqrt{x+2}$. Find $f'(x)$ directly from the definition of the derivative.

Solution: From the definition

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}\right)\left(\frac{\sqrt{(x+h)+2} + \sqrt{x+2}}{\sqrt{(x+h)+2} + \sqrt{x+2}}\right)$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+h)+\cancel{2} - (\cancel{x}+\cancel{2})}{h\left(\sqrt{(x+h)+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{(x+h)+2} + \sqrt{x+2}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{(x+h)+2} + \sqrt{x+2}}$$

$= \dfrac{1}{2\sqrt{x+2}}$. We check that is is right using derivative rules.

Question 7. A related rates problem, I just did these problems to death. See for example, the ladder question from last week (which was not done in class).

MATH 1500 Feb 26

Lecture 20 midterm review

Question 1. Find the limits, if they exist.

(a) $\lim_{x \to 5} \frac{x^3 - 5x^2}{x^2 - 25}$. Plugging $x = 5$ gives $\frac{0}{0}$, so we have to factor out an $(x-5)$.

Factoring: $x^2 - 25 = (x-5)(x+5)$.

$x^3 - 5x^2 = x(x^2 - 5x) = x^2(x-5)$.

$$= \lim_{x \to 5} \frac{x^2\cancel{(x-5)}}{\cancel{(x-5)}(x+5)} = \lim_{x \to 5} \frac{x^2}{x+5} = \frac{25}{5+5} = \frac{25}{10} = \frac{5}{2}.$$

(b) $\lim_{x \to \infty} \frac{\sqrt{4x^2 - x}}{3x - 1}$ Multiply top and bottom by $\frac{1}{x}$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}\sqrt{4x^2 - x}}{\frac{1}{x}(3x-1)} = \lim_{x \to \infty} \frac{\sqrt{\frac{1}{x^2}} \cdot \sqrt{4x^2 - x}}{\frac{1}{x}(3x-1)}$$

$$= \lim_{x \to \infty} \frac{\sqrt{4 - \cancel{\frac{1}{x}}^{\to 0}}}{3 - \cancel{\frac{1}{x}}^{\to 0}} = \frac{\sqrt{4}}{3} = \frac{2}{3}.$$

Question 2: Find $f'(x)$.

(a) $f(x) = \pi^{\frac{1}{3}} + 3x^3 - \frac{2}{x^3} + e^{\sec x}$.

$$= 0 + 3 \cdot 3x^2 - 2 \cdot (-3)x^{-4} + e^{\sec x} \cdot \sec x \tan x.$$

$$= 9x^2 + \frac{6}{x^4} + \sec x \tan x \, e^{\sec x}.$$

(b) $f(x) = (\sin^2 x + \sin x^2)(x^3 + \cos(3x))$

So
$$f'(x) = (\sin^2 x + \sin x^2)'(x^3 + \cos(3x)) + (\sin^2 x + \sin x^2)(x^3 + \cos(3x))'$$
$$= (2\sin x \cos x + \cos(x^2)\cdot 2x)(x^3 + \cos(3x))$$
$$+ (\sin^2 x + \sin x^2)(3x^2 + 3\cdot(-\sin(3x))).$$

(c) $f(x) = \dfrac{(3x+2)^{20}}{x^2 + 2x.}$ $\quad \left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$

$$= \frac{((3x+2)^{20})'(x^2+2x) - (x^2+2x)'(3x+2)^{20}}{(x^2+2x)^2}$$
$$= \frac{(20\cdot(3x+2)^{19}\cdot 3)(x^2+2x) - (2x+2)(3x+2)^{20}}{(x^2+2x)^2}$$

Question 3.

Find the value of a if
$$f(x) = \begin{cases} ax^2 - 3 & x \geq 2 \\ 6x - a & x < 2 \end{cases}$$
is continuous at $x = 2$. Use limits.

Solution: If $f(x)$ is continuous then left and right limits are equal. Therefore we calculate
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2 - 3 = a(2)^2 - 3 = 4a - 3.$$

on the other hand,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 6x - a = 6(2) - a = 12 - a.$$

Therefore we need $12 - a = 4a - 3$

$$\Rightarrow 5a = 15$$

$$\Rightarrow a = 3.$$

<u>Question 4</u>: Find y''' if $y = 5x^3 - 6x^2 + 6$.

<u>Solution</u>: Using the power rule repeatedly,

$$y' = 5(3x^2) - 6(2x) + 0 = 15x^2 - 12x$$

$$y'' = 15(2x) - 12(1) = 30x - 12$$

$$y''' = 30$$

<u>Question 5</u>:

Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{x+5}$.

<u>Solution</u>

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)+5} - \sqrt{x+5}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{(x+h)+5} - \sqrt{x+5}\right)}{h} \frac{\left(\sqrt{(x+h)+5} + \sqrt{x+5}\right)}{\left(\sqrt{(x+h)+5} + \sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{(x+h)+5 - (x+5)}{h\left(\sqrt{(x+h)+5} + \sqrt{x+5}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{(x+h)+5}+\sqrt{x+5}\right)}$$

$$= \frac{1}{\sqrt{(x+0)+5}+\sqrt{x+5}} = \frac{1}{2\sqrt{x+5}}.$$

<u>Question 6</u>: Prove the following theorem:

$$(fg)' = f'g + g'f \quad \text{(assuming } f', g' \text{ exist).}$$

<u>Solution</u>: I showed a trick to get the final answer. It goes like this:

$$(f(x)g(x))' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

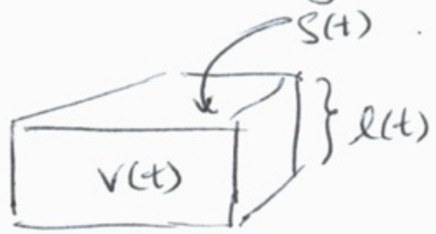
$$= f(x)g'(x) + g(x)f'(x).$$

<u>Question 7</u>. Find the equation of the line tangent to $2x^2y + xy^3 - 3x = 4$ at $(2, 1)$.

Question 8:

The surface area of a cube is increasing at $4 m^2/s$. What is the rate of increase of the volumen when the length of a side is 10?

Solution:



Name the side length $l(t)$, the surface area $S(t)$ and volume $V(t)$. Then.

$$S(t) = 6 \cdot (l(t))^2$$

and $V(t) = (l(t))^3$.

So we get. $$\frac{dS}{dt} = 6 \cdot 2\, l(t) \frac{dl}{dt} \quad (1)$$

and $$\frac{dV}{dt} = 3(l(t))^2 \cdot \frac{dl}{dt}. \quad (2)$$

From (1) we get $4 = 6 \cdot 2 \cdot 10 \dfrac{dl}{dt}$

so that $\dfrac{dl}{dt} = \dfrac{4}{120} = \dfrac{1}{30}$ when $l(t) = 10$.

Therefore $$\frac{dV}{dt} = 3(10)^2 \cdot \frac{1}{30} = \frac{300}{30} = 10 m^3/s$$

when $l(t) = 10$.

Solution: We use implicit differentiation.

$$(2x^2y + xy^3)' - (3x)' = (4)'$$

Then we have a couple product rules:

$$(2x^2y)' = (2x^2)'y + 2x^2y'$$
$$= 4xy + 2x^2y'$$

and

$$(xy^3)' = (x)'y^3 + x(y^3)'$$
$$= y^3 + x3y^2y'$$

So we get

$$4xy + 2x^2y' + y^3 + x3y^2y' - 3 = 0.$$

and $\qquad 2x^2y' + x3y^2y' = 3 - y^3 - 4xy$

$$\Rightarrow y' = \frac{3 - y^3 - 4xy}{2x^2 + x3y^2}.$$

Then set $x=2$, $y=1$.

$$y' = \frac{3 - 1^3 - 4(2)(1)}{2(2)^2 + (2)\cdot 3(1)^2} = \frac{3-1-8}{8+6} = \frac{-6}{14} = \frac{-3}{7}.$$

So $y = \frac{-3}{7}x + b$, with b chosen so that

$$1 = \frac{-3}{7}(2) + b \Rightarrow b = 1 + \frac{6}{7} = \frac{13}{7}.$$

So $y = \frac{-3}{7}x + \frac{13}{7}$.

MATH 1500 Calculus. February 28

§1.6 Lecture 21

Questions: 35-42, 51-58, 63-72.

If $f(x)$ is a function, then recall that

- the domain of $f(x)$ is all numbers x that can be plugged into $f(x)$.
- the range of $f(x)$ is all numbers $f(x)$ that you can get as output from the function $f(x)$.

A function $f(x)$ is called one-to-one if $f(x) = f(y)$ means that $x = y$. In other words, $f(x)$ doesn't take on the same value twice.

Example: Show that $f(x) = 6x - 1$ is one-to-one.

Solution: Suppose that f takes on the same value twice, say $f(x) = f(y)$. Then this means

$$\begin{aligned} f(x) = f(y) &\Rightarrow 6x - 1 = 6y - 1 \\ &\Rightarrow 6x = 6y \\ &\Rightarrow x = y. \end{aligned}$$

So $f(x) = f(y) \Rightarrow x = y$, and $f(x)$ is 1-to-1.

Given a function $f(x)$, if it is one-to-one we can use it to make a new function $f^{-1}(x)$.

If $f(x)$ has domain A and range B, then

$f^{-1}(x)$ has domain B and range A, and its formula is $f^{-1}(y) = x \iff f(x) = y$

<u>Example</u>: (How to solve for an inverse function).

1. Write $f(x)=y$.
2. Solve for x.
3. Interchange y and x in the final equation.

<u>E.g.</u> If $f(x) = \dfrac{x}{7x-1}$, what is $f^{-1}(x)$?

<u>Solution</u>: Solve $y = \dfrac{x}{7x-1}$ for x. We get

$$\Rightarrow y(7x-1) = x$$

$$\Rightarrow 7xy - y = x$$

$$\Rightarrow -y = x - 7xy = x(1-7y)$$

$$\Rightarrow x = \frac{-y}{1-7y}.$$

Interchange x and y. So $y = \dfrac{-x}{1-7x}$, here 'y' represents $f^{-1}(x)$. So $f^{-1}(x) = \dfrac{-x}{1-7x}$.

The key property of an inverse function is that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. For example, we just found $f(x) = \dfrac{x}{7x-1}$ and $f^{-1}(x) = \dfrac{-x}{1-7x}$. So

$$f\left(f^{-1}(x)\right) = \frac{\frac{-x}{1-7x}}{7\left(\frac{-x}{1-\cancel{7}x}\right)-1} = \frac{\frac{-x}{1-7x}}{\frac{-7x}{1-7x}-1}$$

$$= \frac{\frac{-x}{1-7x}}{\frac{-7x-(1-7x)}{1-7x}}$$

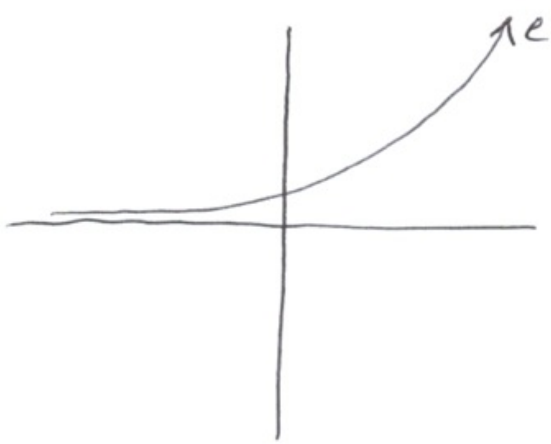
$$= \frac{\frac{-x}{1-7x}}{\frac{-1}{1-7x}} = x$$

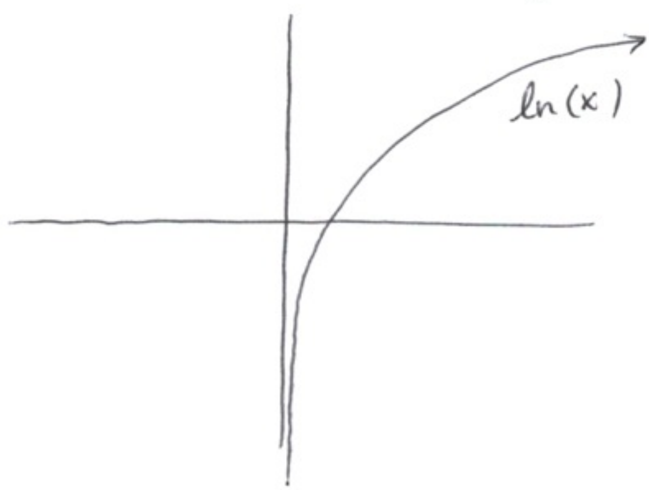
(after cancelling)

We can also check that $f^{-1}(f(x)) = x$, but the calculations are very similar.

Fact: In general, you get the graph of $f^{-1}(x)$ from the graph of $f(x)$ by reflecting in the line $y=x$. (This can also help you check your formula for $f^{-1}(x)$ once you get good at graphing).

Example: The graph of e^x is on the left. It's inverse function is called $\ln(x)$ and is on the right.





In general, $\log_a(x)$ is the inverse of the function $f(x) = a^x$. The function $\ln(x)$ is: $\ln(x) = \log_e(x)$.

What number is $\log_a(x)$? The number $\log_a(X)$ is the answer to the question: To what power must a be raised in order to get x?

So, e.g. if $x>0$ then

$$\log_{10}(100) = 2, \quad \log_3(27) = 3, \quad \log_2\left(\tfrac{1}{32}\right) = \log_2(2^{-5}) = -5.$$

The equations $f(f^{-1}(x)) = x$ and $f^{-1}(f(x))$ become

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a(x)} = x.$$

<u>Logarithm Rules</u>. <u>Exp rule that matches</u>

① $\log_a(xy) = \log_a x + \log_a y \qquad (a^x \cdot a^y = a^{x+y})$

② $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y \qquad (a^x/a^y = a^{x-y})$

③ $\log_a(x^r) = r\log_a x \qquad ((a^x)^r = a^{xr})$

<u>Example</u>: What is the exact value of $2\log_4(8)$?

<u>Solution</u>: Using logarithm rules,

$$2\log_4(8) = \log_4(8^2) = \log_4(64) = \log_4(4^3) = 3.$$

We can use logarithms to solve for variables that appear in the exponent.

<u>Example</u>: Solve for x:

$$e^{10x^2-1} = 5.$$

<u>Solution</u>: Take $\ln$ of both sides, and use the fact that $\ln(e^y) = y$ for any y, because e^x and $\ln(x)$ are inverse to one another.

$$\ln(e^{10x^2-1}) = \ln(5)$$

$$\Rightarrow 10x^2 - 1 = \ln(5)$$

$$\Rightarrow 10x^2 = \ln(5) + 1$$

$$\Rightarrow x^2 = \frac{\ln(5)+1}{10}, \quad \text{so } x = \pm\sqrt{\frac{\ln(5)+1}{10}}.$$

<u>Example</u>: Solve for x:

$$\ln(x^3 + 1) = 10.$$

<u>Solution</u>: Here, we take exponents of both sides and use $e^{\ln(y)} = y$ for every y. Then we get

$$e^{\ln(x^3+1)} = e^{10}$$

$$\Rightarrow x^3 + 1 = e^{10}$$

$$x = \sqrt[3]{e^{10} - 1}.$$

Last, it is important to note that we don't need to work with logarithms in different bases. If we want to talk about $\log_a(x)$, we can write it in terms of $\ln(x)$:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Why is this true?

If $y = \log_a(x)$, then this means

$a^y = x$, so taking ln of both sides:

$$\ln(a^y) = \ln(x)$$

$$\Rightarrow y\ln(a) = \ln(x)$$

$$\Rightarrow y = \frac{\ln(x)}{\ln(a)}.$$

This is the reason that we always work with e^x and $\ln(x)$, because this formula allows us to change to a different base if necessary.