MATH 1500 Calc. January 27
Lecture 10

Last day we saw the derivative of $f(x)$ at $x=a$, which is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x\to a} \frac{f(x) - f(a)}{x-a}.$$

The number $f'(a)$ is also the slope of the tangent line at $x=a$.

<u>Example</u>: If $f(x) = \frac{1}{x^2}$, find $f'(2)$.

<u>Solution</u>: The formula is:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{h}$$

<u>The top is</u>: $\frac{1}{(2+h)^2} - \frac{1}{2^2} = \frac{1}{4+4h+h^2} - \frac{1}{4}$

$$= \frac{4-(4+4h+h^2)}{16+16h+4h^2} = \frac{-4h-h^2}{16+16h+4h^2}.$$

So the limit is

$$\lim_{h\to 0} \frac{(-4-h)\cancel{h}}{(16+16h+4h^2)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-4-h}{16+16h+4h^2} = \frac{-4}{16} = \frac{-1}{4}.$$

For each 'a' value, we get a different number from the limit $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

So we can make a new function called the derivative of f

Definition of derivative:

The derivative of $f(x)$ is a function called $f'(x)$ whose formula is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative $f'(x)$ is interpreted these ways:

(i) The number $f'(a)$ is the slope of the tangent line to $f(x)$ at $x=a$

(ii) The number $f'(a)$ is the rate of change of $f(x)$ at $x=a$.

Example: If $f(x) = 2x^2+3$, what is $f'(x)$?

Solution: By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

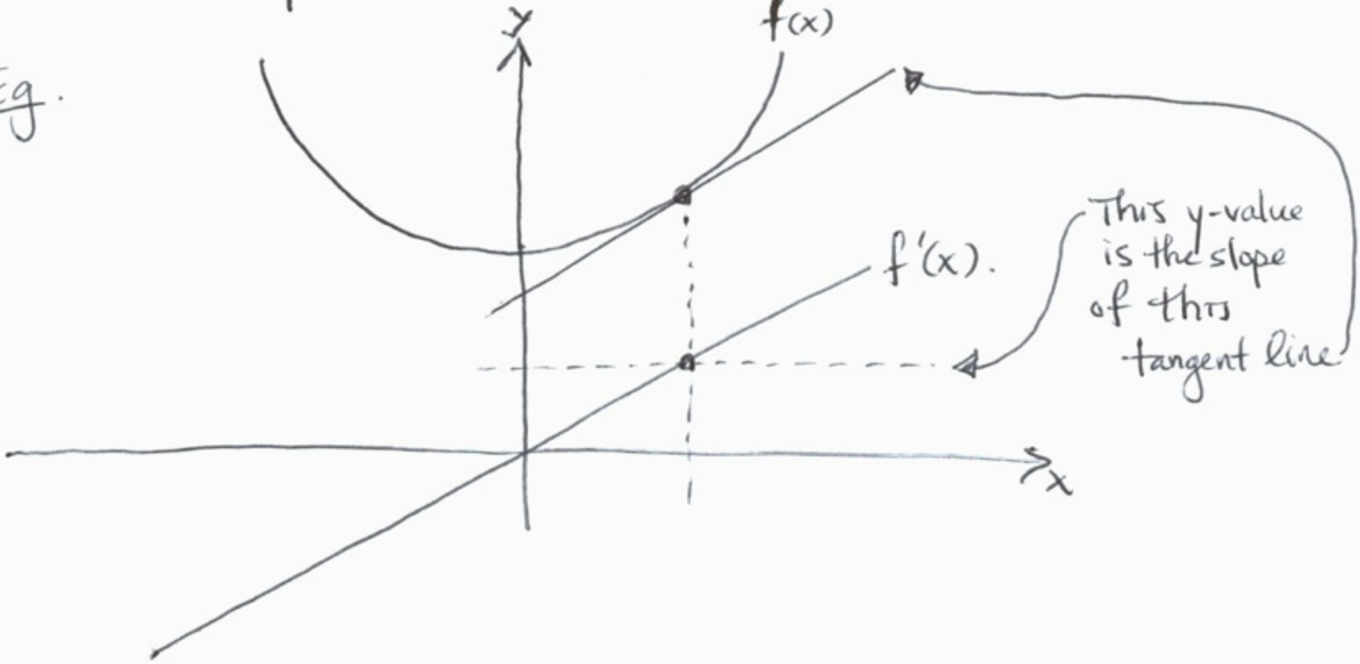
$$= \lim_{h \to 0} \frac{2(x+h)^2 + 3 - (2x^2+3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2(x^2}+2xh+h^2)\cancel{+3}-\cancel{2x^2}\cancel{-3}}{h}$$

$$= \lim_{h \to 0} \frac{2(2xh + h^2)}{h} = \lim_{h \to 0} 2(2x + h) = \cancel{2x}\ 4x$$

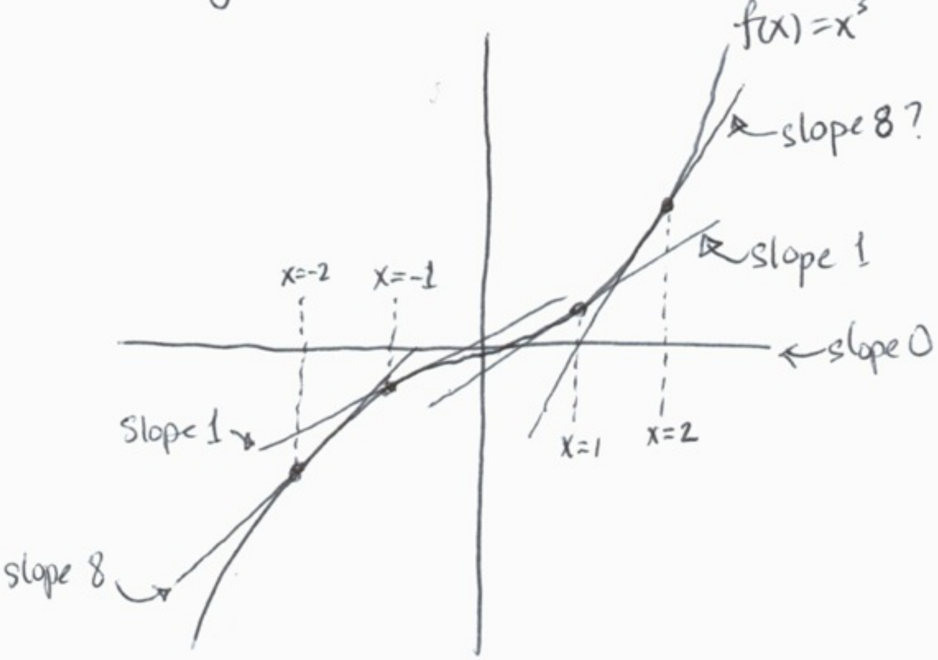
So $f'(x) = 4x$.

Eg.



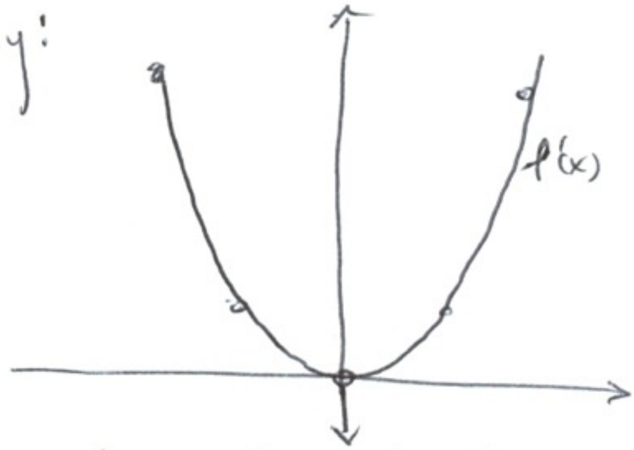
<u>Example</u>: Estimate from the graph of $f(x) = x^3$ what the graph of $f'(x)$ should look like.

<u>Solution</u>: We draw $f(x) = x^3$, and use estimated tangent line slopes to plot $f'(x)$.



x	$f'(x)$
-2	8 ?
-1	1 ?
0	0
1	1 ?
2	8 ?

The graph of $f'(x)$ is roughly:



This is roughly correct, the exact equation for $f'(x)$ is $f'(x) = 3x^2$.

Terminology: A function $f(x)$ is called differentiable at $x=a$ if

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

Otherwise it's called not differentiable.

Example: Show that $f(x) = |x|$ is not differentiable at $a=0$.

Solution: This means we have to check if

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

exists. We check existence of limits by doing left/right limits. So:

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = \lim_{h \to 0} -1 = -1.$$

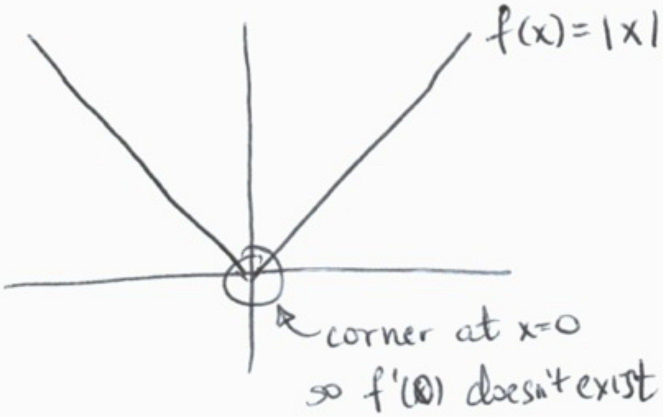
$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = \lim_{h \to 0} 1 = 1.$$

Since left and right limits are different,

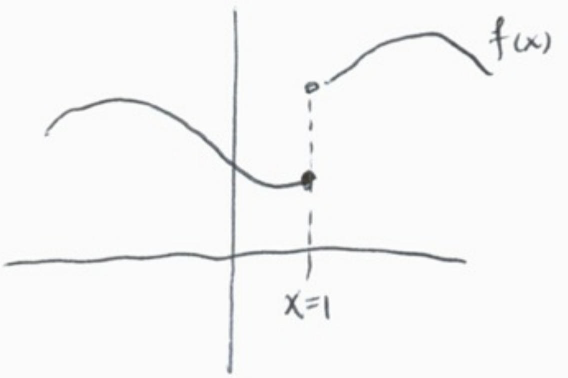
$$f'(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$ doesn't exist, i.e. f is not

<u>differentiable</u> at $x=0$.

<u>Fact</u>: In general, if a function $f(x)$ has a 'corner' then $f'(x)$ doesn't exist at the corner, eg.

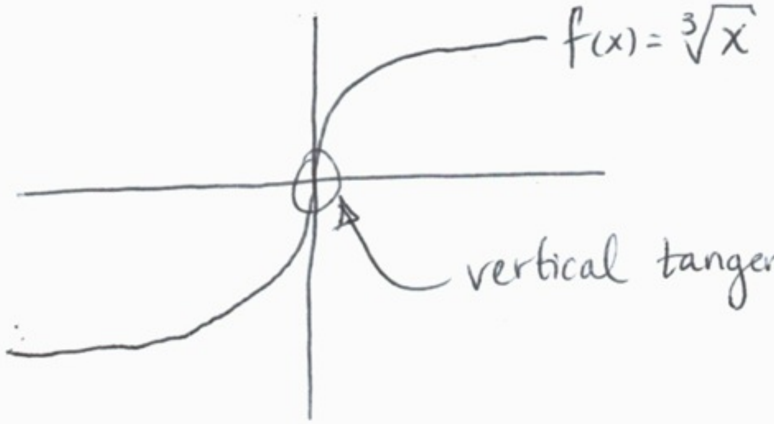


Or, if $f(x)$ has a 'jump':



Since there is a jump at $x=1$, $f'(1)$ does not exist.

Or, if $f(x)$ has a vertical tangent line at some point:



vertical tangent at $x=0$ so $f'(0)$ doesn't exist.

Other notation:

In place of $f'(x)$, we also write $\frac{df}{dx}$, or if

we have $y=f(x)$ sometimes $\frac{dy}{dx}$.

So for example we saw if $f(x)=2x^2+3$, then $f'(x)=4x$.
We could also write $\frac{df}{dx}=4x$ or $f(x)=y$ and

$\frac{dy}{dx}=4x$.

In place of $f'(3)=4\cdot 3=12$ we write

$$\left.\frac{df}{dx}\right|_{x=3}=4.3=12.$$

Example: If $y=\cos(x)$, what is $\frac{dy}{dx}$?

Solution: $\frac{dy}{dx}$ is:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)}{h}-\lim_{h\to 0}\frac{\sin(x)\sin(h)}{h}$$

we can factor out cos(x) from the first limit, sin(x) from the second, and get:

$$= \cos(x) \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}} - \sin(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}.$$

These limits are classics that you can look up in any textbook.

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

So

$$\frac{dy}{dx} = \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x).$$

ie. $\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$

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Lecture 11.

Last day we ended with new notation:

Write $\frac{df}{dx}$ in place of $f'(x)$ (or $\frac{dy}{dx}$)

and $\left.\frac{df}{dx}\right|_{x=a}$ in place of $f'(a)$, or sometimes

$\frac{df}{dx}(a)$, but technically this is not correct.

"Leibniz notation"

Example: If $y = \sin(x)$, show that $\frac{dy}{dx} = \cos(x)$.

Solution: The formula for the derivative is

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Recall the trig formula: $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x) \underbrace{\lim_{h \to 0} \frac{\cos(h)-1}{h}} + \cos(x) \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}.$$

These are both classic limits, whose values are hard to determine (can be done using squeeze theorem)

Their values are

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

So our formula becomes

$$\frac{dy}{dx} = \sin(x)(0) + \cos(x)(1) = \cos(x).$$

Here is one of the only proofs we will do in class.

$$\text{If } \underbrace{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}_{f \text{ is differentiable at } a} \text{ exists, then } \underbrace{\lim_{x\to a} f(x) = f(a)}_{f \text{ is continuous at } a}.$$

Here is how we show differentiable at $x=a$ implies continuous at $x=a$.

We assume $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ exists.

This is the same as assuming $\displaystyle f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

Then $$f(x) - f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a)$$

so
$$\begin{aligned}
\lim_{x\to a}\left(f(x)-f(a)\right) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) \cdot \lim_{x\to a}(x-a) \\
&= f'(a)\cdot 0 \quad \left(\text{By assumption } f'(a) \text{ exists}\right) \\
&= 0
\end{aligned}$$

Carefully examining this equation:

$$\lim_{x \to a} \left(f(x) - f(a) \right) = 0$$

(Limit laws) $$\lim_{x \to a} f(x) - \lim_{x \to a} f(a) = 0$$

$$\lim_{x \to a} f(x) - f(a) = 0 \qquad \text{(Limit of a constant)}$$

So, $\lim_{x \to a} f(x) = f(a)$, i.e. f is continuous at $x = a$.

Higher derivatives:

Not surprisingly, you can take derivatives of derivatives.

Example: Compute the first, second, and third derivative of $f(x) = \cancel{x^{10}}, x^3$.

Solution:

We already saw that $f'(x) = 3x^2$.

Then the second derivative is:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3(2hx + h^2)}{h} = \lim_{h \to 0} 3(2x + h) = 6x.$$

The third derivative is $f'''(x)$ or $f^{(3)}(x)$, and it is:

$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{6}(\cancel{x}+h) - \cancel{6x}}{h}$$

$$= \lim_{h \to 0} \frac{6h}{h} = 6.$$

So the first, second and third derivatives are $3x^2$, $6x$, and 6.

If the first derivative is the rate of change, the second is the rate of change of the rate of change. This is commonly called accelaration:

$$d(t) \xrightarrow{\text{differentiate}} d'(t) = v(t) \xrightarrow{\text{differentiate}} d''(t) = a(t)$$

original function | velocity | acceleration.

In our $\frac{d}{dx}$ notation, higher derivatives are written this way:

$f''(x)$ is $\frac{d^2f}{dx^2}$

$f'''(x)$ is $f^{(3)}(x)$ or $\frac{d^3f}{dx^3}$

$f^{(4)}(x)$ is $\frac{d^4f}{dx^4}$, etc.

Chapter 3. Rules for calculating derivatives.

Rule 1 If $f(x) = c$ is a constant function, then $f'(x) = 0$. Or in other notation:

$$\frac{d}{dx}(c) = 0.$$

Rule 2: If $f(x) = x^n$ for any number n, then $f'(x) = nx^{n-1}$, or

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Rule 3: If $f(x)$ and $g(x)$ are differentiable and a, b are numbers, then

$$\left(af(x) + bg(x)\right)' = af'(x) + bg'(x) \quad \underline{\text{or}}$$

$$\frac{d}{dx}\left(af(x) + bg(x)\right) = a\frac{df}{dx} + b\frac{dg}{dx}.$$

Example: If $f(x) = \sqrt[3]{x}$, what is $f'(x)$?

Solution: $f(x) = \sqrt[3]{x}$ can be written as

$$f(x) = x^{\frac{1}{3}}, \text{ so } f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$$

$$f'(x) = \frac{1}{3}\frac{1}{x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}.$$

Example: If $f(x) = 3x^5 - 4x^2 + 5$, what is $\frac{df}{dx}$?

Solution: Using the rules, we get:

$$\frac{df}{dx} = \frac{d}{dx}(3x^5 - 4x^2 + 5)$$

$$= 3\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^2) + \frac{d}{dx}5$$

$$= 3(5x^4) - 4(2x) + 0 = 15x^4 - 8x.$$

Example: Where is the tangent line to $y = x^3 - 6x^2 + 11x - 6$ horizontal?

Solution: The tangent line is horizontal when its slope is zero. So we solve $\frac{dy}{dx} = 0$. Using the rules:

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) - 6\frac{d}{dx}(x^2) + 11\frac{d}{dx}x - \cancelto{0}{\frac{d}{dx}6}$$

$$= 3x^2 - 12x + 11$$

Solving $\frac{dy}{dx} = 0$, we get $3x^2 - 12x + 11$, and the quadratic equation gives:

$$x_1, x_2 = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(11)}}{6} = \frac{12 \pm \sqrt{144 - 132}}{6}$$

$$= \frac{12 \pm \sqrt{12}}{6} = \frac{6 \pm \sqrt{3}}{3}.$$

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Lecture 12.

§3.1: Derivative rules.

The first three rules are:

1. $\frac{d}{dx}(c) = 0$ or if $f(x) = c$, then $f'(x) = 0$.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$, or if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ (here, n is any number).

3. If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}(af(x) + bg(x)) = a\frac{df}{dx} + b\frac{dg}{dx}, \text{ or}$$

$$(af(x) + bg(x))' = af'(x) + bg'(x).$$

Example: What is the derivative of $f(x) = x^6 - 5x^5 + \sqrt{x}$?

Solution: Note that $\sqrt{x} = x^{\frac{1}{2}}$, so we can use the rules above:

$$\frac{df}{dx} = \frac{d}{dx}(x^6 - 5x^5 + x^{\frac{1}{2}}) = \frac{d}{dx}x^6 - 5\frac{d}{dx}x^5 + \frac{d}{dx}x^{\frac{1}{2}}$$

$$= 6x^5 - 5(5x^4) + \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= 6x^5 - 25x^4 + \frac{1}{2}x^{-\frac{1}{2}}$$

$$= 6x^5 - 25x^4 + \frac{1}{2\sqrt{x}}.$$

Example: What is the equation of the line tangent to $f(x) = x^3 - 1$ at the point $(2,7)$?

Solution: By the derivative rules,

$f'(x) = 3x^2 - 0 = 3x^2$, so the slope of the tangent line is $f'(2) = 3(2)^2 = 3\cdot 4 = 12$.

The equation of the tangent line is therefore $y = mx+b$ with $m = 12$ and 'b' chosen so that the line passes through $(2,7)$. So to find b:

$$7 = 12(2) + b \Rightarrow b = 7 - 24 = -17.$$

The tangent line equation is

$$y = 12x - 17.$$

Question: Where do the derivative rules come from?

Ans: From the limit rules and definition of derivative.

Example: Show that $\frac{d}{dx}(cf) = c\frac{df}{dx}$ (i.e. $(cf)' = cf'$). if $f(x)$ is differentiable.

Solution: Here, we have:

$$\begin{aligned}
\frac{d}{dx}(cf(x)) &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h\to 0} \frac{c(f(x+h) - f(x))}{h} \quad (\text{limit rules}) \\
&= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= c f'(x) = c\frac{df}{dx}.
\end{aligned}$$

Example: Show $(f+g)' = f' + g'$ if $f(x)$ and $g(x)$ are both differentiable.

Solution: By definition of the derivative,

$$(f(x)+g(x))' = \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right].$$

limit rules.

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

So these last two examples combined show why Rule 3 works: $(af+bg)' = af' + bg'$.

Example: Suppose $f(x) = a^x$, 'a' is any number? What is $f'(0)$? What is $f'(x)$?

Solution: The formula is:

$$f'(0) = \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \underbrace{\lim_{h\to 0} \frac{a^h - 1}{h}}_{\text{who knows what number this is. On to the next part....}} \quad (a^0 = 1).$$

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

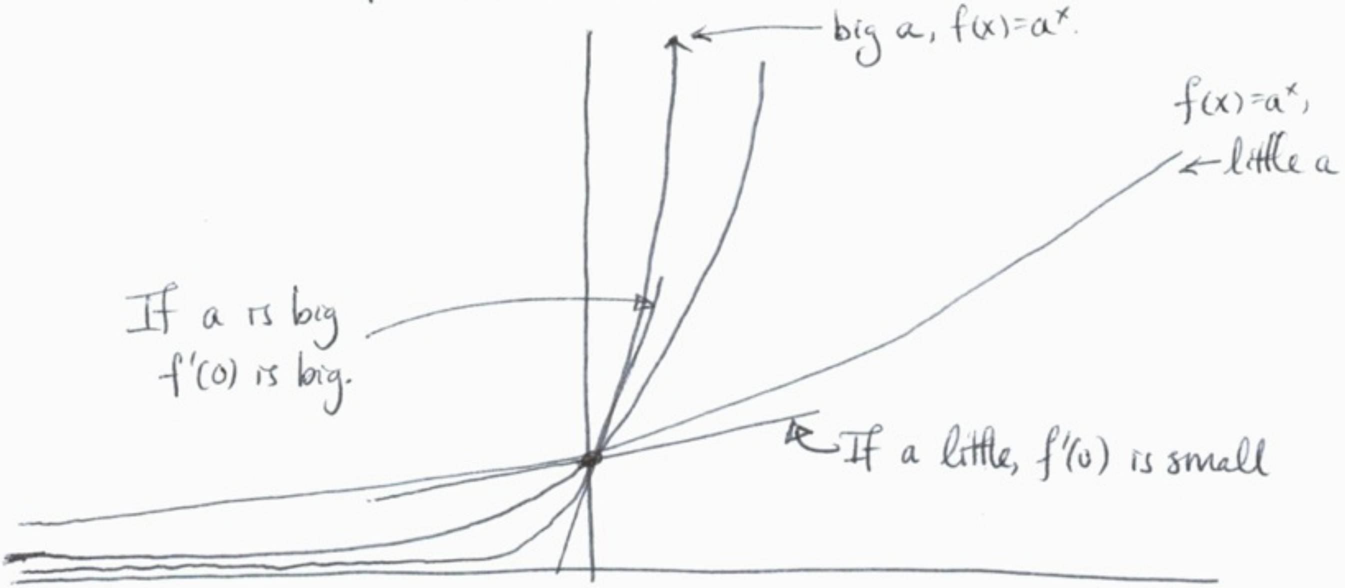
$$= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

So at least we can say $f'(x) = f'(0)\, a^x$

whatever this number is.

What is the number $f'(0)$ if $f(x) = a^x$?

Well, that depends on the value of a:



Ans: $f'(0)$ can be very small by choosing a to be small, or v. big by choosing a to be big.

The best case would be if ~~wh~~ we choose a so that $f'(0) = 1$, then $f'(x) = a^x \cdot 1 = a^x$.

I.e we need to choose a so that

$$f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h} = 1.$$

Definition: The number $e \approx 2.718\ldots$ satisfies

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Therefore, if $f(x) = e^x$ then $f'(x) = e^x \cdot f'(0) = e^x \cdot 1 = e^x$.

I.e.

$$\boxed{\underline{\text{Rule 4}}: \quad \frac{d}{dx}(e^x) = e^x}$$

Example: What is the point on the curve $y = e^x$ where the tangent line has slope e^2?

Solution: The number e^2 looks awful, but thanks to the magical properties of e, everything works!

Here, $f'(x) = e^x$, so at $x = 2$ the tangent line has slope $f'(2) = e^2$, done.

Example: Where does the function $f(x) = e^x - x$ have a horizontal tangent line?

Solution: The tangent line is horizontal when $f'(x) = 0$. So we calculate:

$$f'(x) = (e^x)' - (x)'$$
$$= e^x - 1$$

Then set $e^x - 1 = 0$. So $e^x = 1$

$$\Rightarrow x = 0 \quad \left(\text{since } a^0 = 1 \text{ for any } a\right).$$

Therefore the tangent line is horizontal at $x=0$.

Example (55).

Find the equation of a line tangent to $f(x)=x\sqrt{x}$ and parallel to $y=1+3x$.

Solution: Tangent to $f(x)=x\sqrt{x}=x\cdot x^{\frac{1}{2}}=x^{3/2}$ means the slope comes from plugging a number 'a' into the formula

$$f'(x)=\frac{3}{2}x^{3/2-1}=\frac{3}{2}\sqrt{x}.$$

Parallel to $y=1+3x$ means the slope is 3. Therefore

$$f'(a)=\frac{3}{2}\sqrt{a}=3$$

so solve for a:

$$\sqrt{a}=3\cdot\frac{2}{3}=2$$

$$a=4.$$

Thus $f(x)$ has the tangent line of correct slope at $x=4$. The eqn of it is:

$y=mx+b$, $m=4$ passing through $f(4)=4\cdot 2=8$.
$(4,8)$.

$8=3\cdot 4+b$, so $b=-4$.

Thus $y=3x-4$.