Calculus 1500 Jan 13

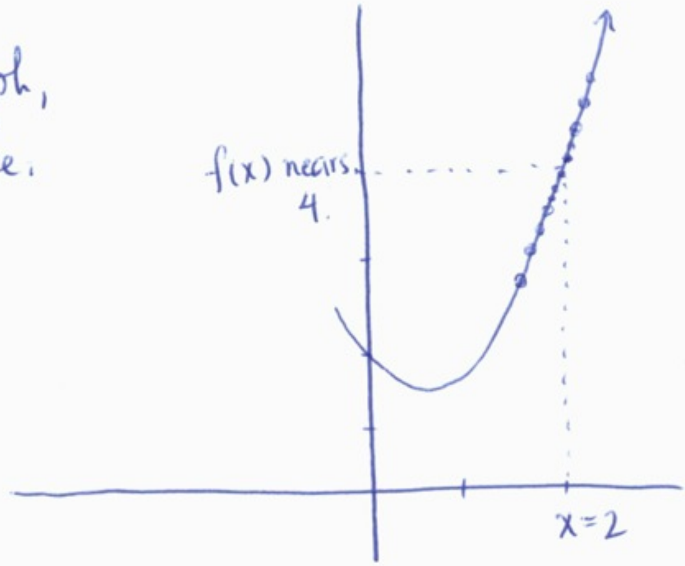
Lecture 4.

§ 2.2 Limits.

Suppose we're graphing the function $f(x) = x^2 - x + 2$, and we make a table of values near $x=2$.

x	$f(x)$
1.9	3.7100
1.95	3.8525
1.99	3.9701
1.999	3.9970
2.001	4.0030
2.01	4.0301
2.05	4.1525
2.1	4.3100

So in the graph, we have.



Literally: As we plug in values of x that are closer and closer to $x=2$, the values of $f(x)$ get closer and closer to $f(x) = 4$. THIS IS A LIMIT!

We write $\lim\limits_{x \to 2} x^2 - x + 2 = 4.$

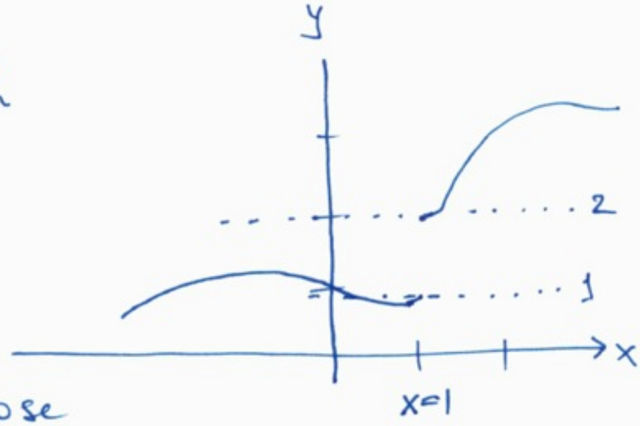
In general, if $f(x)$ is a function that gives numbers closer and closer to L as you plugin x values that are closer and closer to a, write

$$\lim_{x \to a} f(x) = L.$$

Remark: It has to work if you plug in values that are a little bit bigger than a, and a little bit smaller than a (not equal to a).

For example:

If $f(x)$ has graph

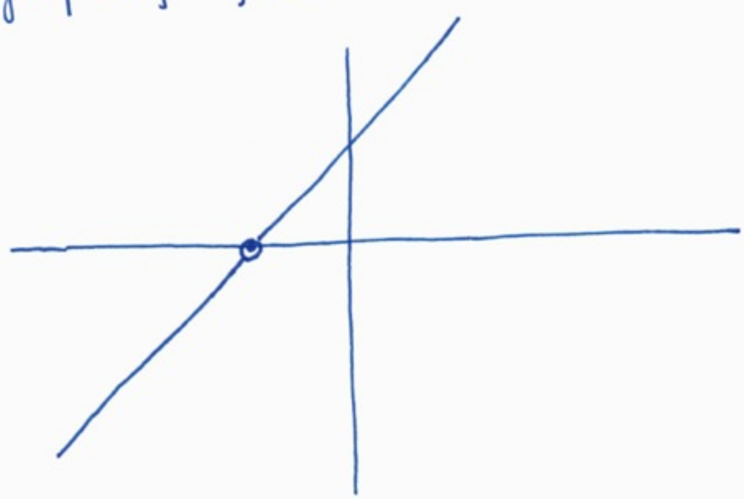


then plugging in x values close to $x=1$ causes a problem: Numbers less than $x=1$ gives $f(x)$ near 1, numbers bigger than $x=1$ gives $f(x)$ near 2. In this case we say $f(x)$ does not have a limit as $x \to 1$.

Example: Suppose $f(x) = \dfrac{x^2+2x+1}{x+1}$. We can factor the top and get $f(x) = \dfrac{(x+1)(x+1)}{(x+1)}$, as long as $x \neq -1$

this means $f(x) = x+1$ (when $x=-1$ we get $f(-1) = \frac{0}{0}$, fail).

So the graph of $f(x)$ is

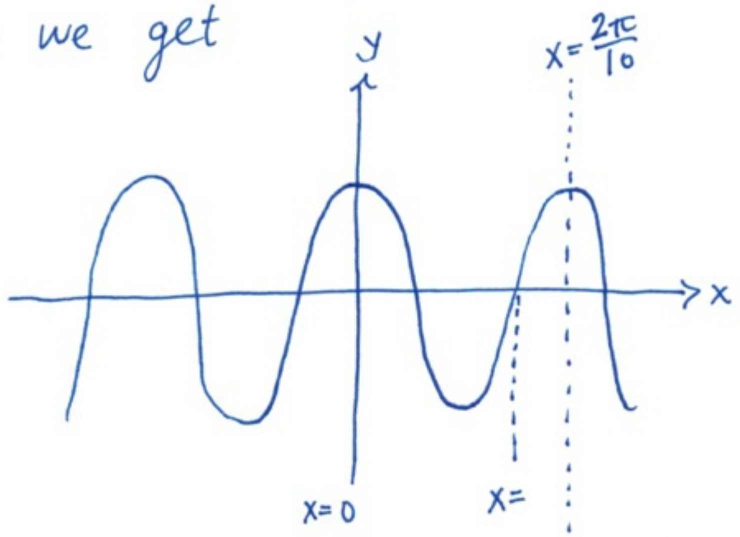


i.e. it looks like the graph of $x+1$, except there is a hole at $x=-1$.

Therefore $\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2+2x+1}{x+1} = 0.$

Example: What is $\lim_{x \to 0} \cos(10x)$?

Solution: The graph of $\cos(10x)$ is the same as the graph of $\cos(x)$, compressed horizontally by a factor of 10. So we get



We can see from the graph that plugging in x-values close to zero gives values of $f(x)$ close to $f(0) = \cos(0) = 1$.

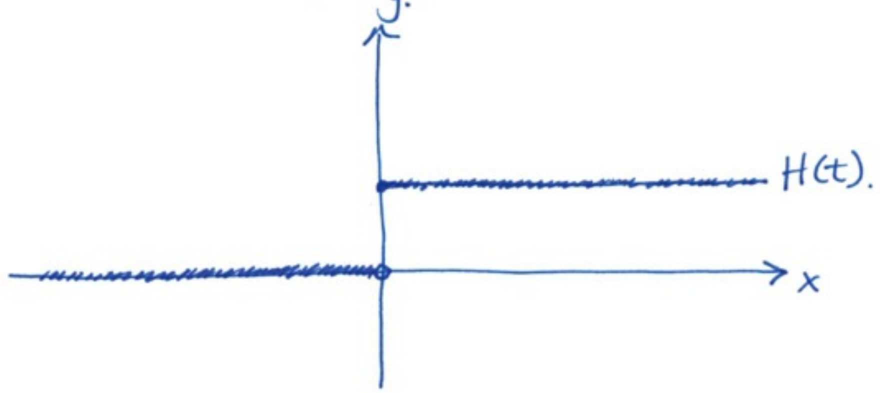
Remark: If f is a "reasonable" function (i.e. it is continuous) then $\lim_{x \to a} f(x) = f(a)$, in other words you get the limit at a by plugging in a.

Other kinds of limits

We can take limits from only one side. For example, if

$$H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t \geq 1 \end{cases}$$

then $H(t)$ has graph



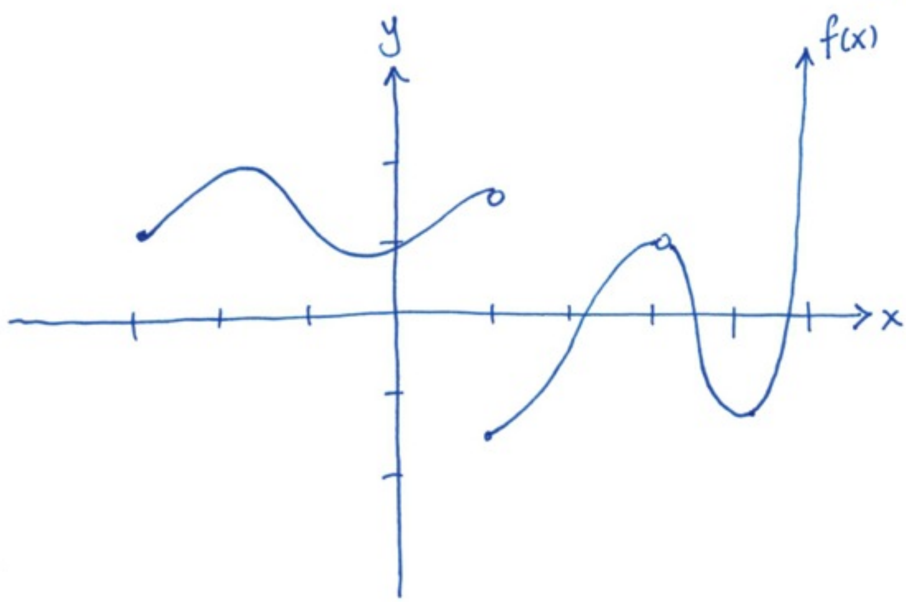
So we can see that coming in to $x=0$ from the left, $H(t)$ is at 0. From the right, $H(t)$ is at 1. We say the left limit is zero, the right limit is one, and write

$$\lim_{t\to 0^-} H(t)=0 \quad \text{and} \quad \lim_{t\to 0^+} H(t)=1.$$

means only values of $t<0$

means only values of $t>0$.

Example: Suppose that $f(x)$ has graph:



Then what are the following limits?

a) $\lim_{x\to 1^+} f(x) = -1.5$

b) $\lim_{x\to 1^-} f(x) = +1.5$

c) $\lim_{x\to 1} f(x)$ does not exist, because without specifying a side of $x=1$ we cannot choose between ± 1.5.

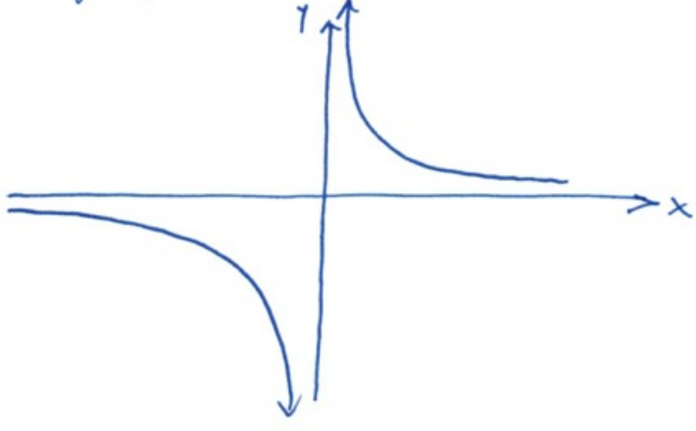
d) $\lim_{x\to 3} f(x) = 1$

e) $\lim_{x\to 5^-} f(x) = +\infty$ (this symbol "∞" is meant to indicate that $f(x)$ gets bigger and bigger as we get closer to $x=5$ from the left).

Remark: If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ are different, we can always say that $\lim_{x\to a} f(x)$ does not exist.

Example: Evaluate $\lim_{x\to 0^-} \frac{1}{x}$ and $\lim_{x\to 0^+} \frac{1}{x}$.

Solution: The graph of $\frac{1}{x}$ is:



As x approaches 0 from the left, $\frac{1}{x}$ becomes huge and negative. So we write:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

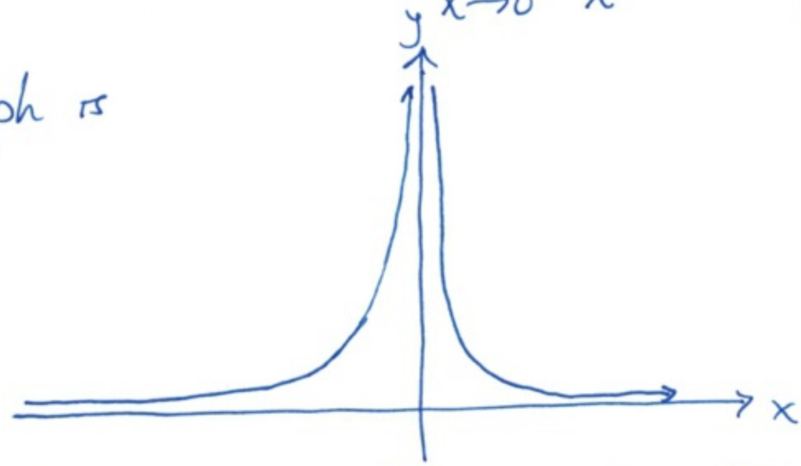
On the other hand, $\lim_{x \to 0^+} \frac{1}{x} = +\infty.$

As in the previous remark, the left and right limits are different at zero, so $\lim_{x \to 0} \frac{1}{x}$ does not exist.

<u>Example</u>: On the other hand, $\lim_{x \to 0} \frac{1}{x^2} = \infty$,

because $\lim_{x \to 0^-} \frac{1}{x^2} = \infty = \lim_{x \to 0^+} \frac{1}{x^2}$.

The graph is



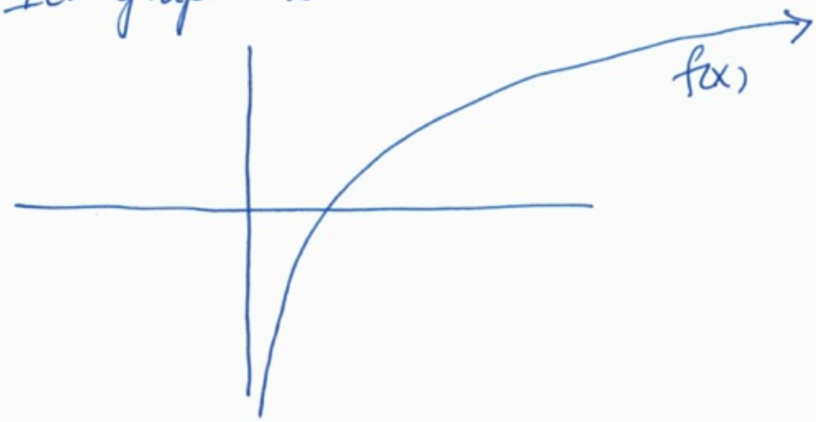
(So $f(x) = \frac{1}{x^2}$ becomes really big and positive on either side of 0).

<u>Definition</u>: The line $x = a$ is called a <u>vertical asymptote</u> of $f(x)$ if one of these is true:

$$\lim_{x \to a} f(x) = \pm\infty, \quad \lim_{x \to a^-} f(x) = \pm\infty, \text{ or } \lim_{x \to a^+} f(x) = \pm\infty.$$

Example: If $f(x) = \frac{1}{(3x-4)^2}$, then as x approaches $\frac{4}{3}$, f(x) approaches "$\frac{1}{0}$" or, it becomes very large. Since the bottom is squared, it becomes very large and positive. So $\lim_{x \to \frac{4}{3}} f(x) = +\infty$, and f(x) has a vertical asymptote at $x = \frac{4}{3}$.

Example: Recall $f(x) = \ln(x)$ is the natural logarithm of x. Its graph is:



Some important limits are:

$$\lim_{x \to 0^+} \ln(x) = -\infty,$$

and the domain of f(x) is $(0, \infty)$. (Not 0, and no negatives!).

MATH 1500 January 15

Lecture 5.

Last class we introduced limits and used graphs / tables of values to find them. Now we introduce rules for calculating $\lim_{x \to a} f(x)$.

First batch of rules:

Suppose c is a constant, and you already know that $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$. Then

① $\lim_{x \to a} \left(f(x) \pm g(x) \right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L_1 \pm L_2$.

② $\lim_{x \to a} \left(c f(x) \right) = c \left(\lim_{x \to a} f(x) \right) = c L_1$.

③ $\lim_{x \to a} f(x) g(x) = \left(\lim_{x \to a} f(x) \right) \left(\lim_{x \to a} g(x) \right) = L_1 L_2$.

④ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$ $\left(\text{as long as } L_2 \neq 0 \right)$.

Example: ~~Last~~

Suppose that $\lim_{x \to 1} f(x) = 4$ and $\lim_{x \to 1} g(x) = -2$.

Calculate $\lim_{x \to 1} \left(5f(x) - 2f(x)g(x)\right)$.

<u>Solution</u>: Using the limit laws, we get:

$$\lim_{x \to 1} \left(5f(x) - 2f(x)g(x)\right) = \lim_{x \to 1} 5f(x) - \lim_{x \to 1} 2f(x)g(x) \quad \text{(Law 1)}.$$

$$= 5 \lim_{x \to 1} f(x) - 2 \lim_{x \to 1} f(x)g(x) \quad \text{(Law 2)}.$$

$$= 5 \lim_{x \to 1} f(x) - 2\left(\lim_{x \to 1} f(x)\right)\left(\lim_{x \to 1} g(x)\right) \quad \text{(Law 3)}$$

$$= 5(4) - 2(4)(-2) = 20 + 16 = 36.$$

With a few more rules, this is all we need to start calculating limits.

(5) $\lim_{x \to a} c = c$.

(6) $\lim_{x \to a} x^p = a^p$, p any power. So in particular since $x^{\frac{1}{2}} = \sqrt{x}$, we have $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ as long as $a > 0$.

(Note: This formula is true as long as a^p is defined, e.g. $p = \frac{1}{2}$ and $a = -1$ doesn't work).

Example: Calculate $\lim_{x \to 2} \frac{x^2+x-1}{3x+5}$.

Solution: Using limit laws:

$$\lim_{x \to 2} \left(\frac{x^2+x-1}{3x+5} \right) = \frac{\lim_{x \to 2} (x^2+x-1)}{\lim_{x \to 2} (3x+5)}$$

$$= \frac{\lim_{x \to 2} x^2 + \lim_{x \to 2} x + \lim_{x \to 2} 1}{3\left(\lim_{x \to 2} x\right) + \lim_{x \to 2} 5}$$

then we use rules 5 and 6 to get actual numbers:
$\lim_{x \to 2} x^2 = 4$, $\lim_{x \to 2} x = 2$ and substitute

$$= \frac{4+2-1}{3(2)+5} = \frac{5}{11}$$

Note that this is the same as simply plugging in $x=2$.

Remark: In general taking a limit is different than plugging in a number, but for polynomials it is the same thing. For rational functions $\left(\text{e.g } \frac{p(x)}{q(x)}\right)$ it is also the same thing, as long as plugging in the number doesn't give division by zero. I.e. we have

$$\lim_{x \to a} p(x) = p(a) \text{ and } \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

as long as $q(a) \neq 0$.

Example: Find $\lim\limits_{x\to 2} \dfrac{(x+3)(x-2)}{(x-2)}$ using limit laws.

Solution: We cannot plug in $x=2$, and we cannot apply the limit laws directly since the bottom has a limit of 0. So the trick is:

$$\lim_{x\to 2} \frac{(x+3)(x-2)}{(x-2)} = \lim_{x\to 2}(x+3) = \lim_{x\to 2} x + \lim_{x\to 2} 3$$

$$= 2+3 = 5.$$

Example: Find $\lim\limits_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$. using limit laws.

Solution: Again we cannot apply the limit laws directly since $\lim\limits_{t\to 0} t^2 = 0$, so the bottom gives a problem.

We do some preliminary algebra to get it to work:

Trick: $$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = 1.$$

$$= \frac{(t^2+\cancel{9}) + \cancel{3\sqrt{t^2+9}} - \cancel{3\sqrt{t^2+9}} - \cancel{9}}{t^2(\sqrt{t^2+9}+3)}$$

$$= \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)} = \frac{1}{\sqrt{t^2+9}+3}$$

Now we can take limits.

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

<u>Example</u>: Show that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

<u>Solution</u>: Remember that $\lim_{x\to a} f(x)$ exists only when $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

In this case, when x is to the left of zero then $|x| = -x$, so

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = \underline{-1}$$

When x is to the right of 0, $|x| = x$ and

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = \underline{1}.$$

Example

If $f(x) = \begin{cases} x^2 + x - 1 & \text{for } x < 2 \\ \sqrt{x^4 + 9} & \text{for } x \geq 2; \end{cases}$

does $\lim_{x \to 4} f(x)$ exist? If so, what is it?

Solution: The rules of limits apply to left and right limits as well. So we calculate:

$$\lim_{x \to 2^-} (x^2 + x - 1) = \lim_{x \to 2^-} x^2 + \lim_{x \to 2^-} x - \lim_{x \to 2^-} 1$$

$$= 2^2 + 2 - 1 = 5$$

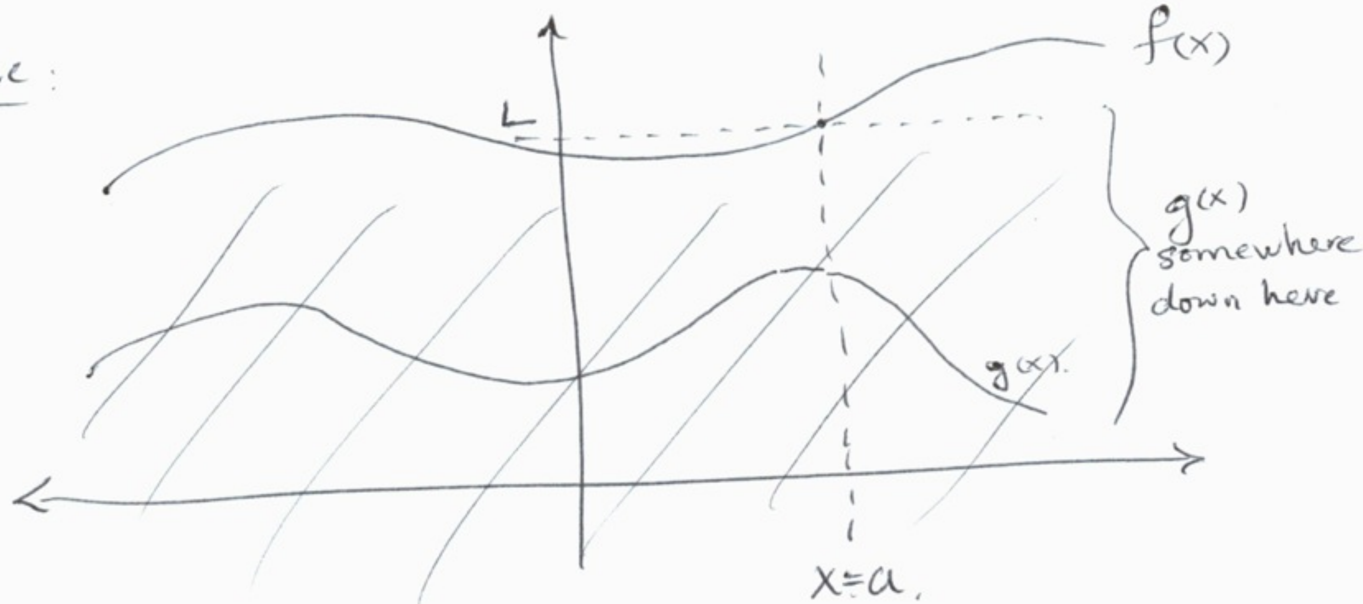
and
$$\lim_{x \to 2^+} \sqrt{x^4 + 9} = \sqrt{\lim_{x \to 2^+} (x^4 + 9)}$$

$$= \sqrt{2^4 + 9} = \sqrt{25} = 5.$$

So since the left and right limits exist and are the same, $\lim_{x \to 2} f(x)$ exists and $\lim_{x \to 2} f(x) = 5$.

__Fact__: If $f(x) \le g(x)$ when x is near a (but not necessarily when $x = a$), then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$, if both limits exist.

__I.e__:



Then as $x \to a$, $g(x)$ must approach something less than $\lim_{x \to a} f(x) = L$.

In general, we have a squeeze theorem.

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Lecture 6.

Last day, we learned some ways of calculating limits aside from plugging in numbers (limit laws).

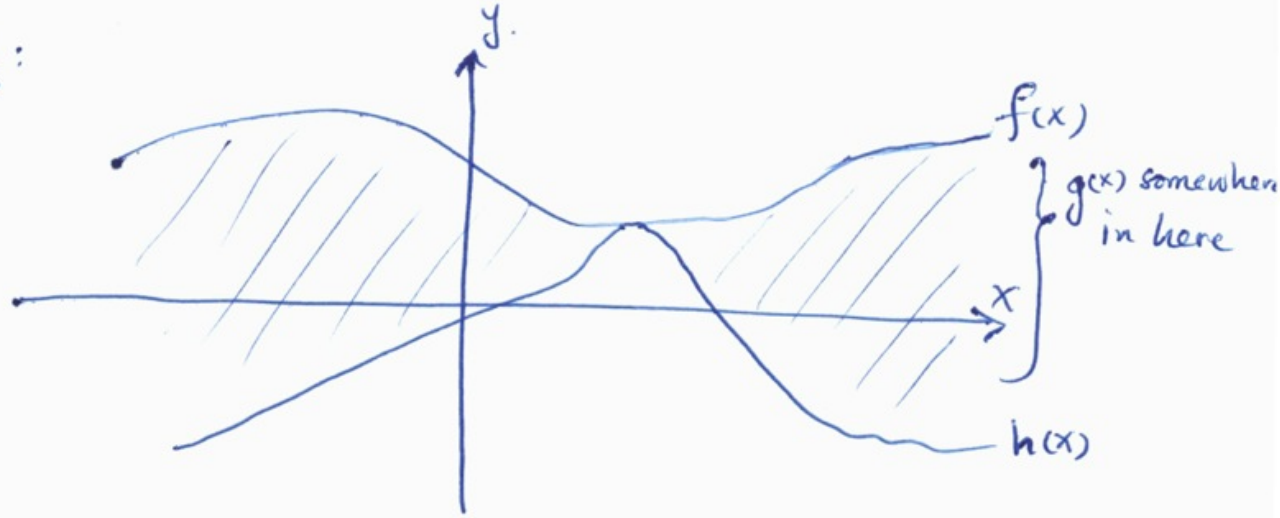
Today, a final trick:

The squeeze theorem: If $f(x) \leq g(x) \leq h(x)$ for x near a (except at a), and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then $\lim_{x\to a} g(x) = L$ as well.

Picture:



Example: What is $\lim_{x\to 0^-} x^3 \cos\left(\frac{2}{x}\right)$?

Solution: Note that we cannot use the product rule:

$$\lim_{x\to 0^-} x^3\cos\left(\frac{2}{x}\right) = \left(\lim_{x\to 0^-} x^3\right)\left(\lim_{x\to 0^-}\cos\left(\frac{2}{x}\right)\right)$$

because $\lim_{x\to 0^-} \cos\left(\frac{2}{x}\right)$ doesn't exist.

pic of $\cos\left(\frac{2}{x}\right)$.

However we can use the squeeze theorem:

First, $\quad -1 \le \cos\left(\frac{2}{x}\right) \le 1 \quad$ (cosine always gives numbers between -1 and 1).

Then we multiply by x^3, since we're coming at 0 <u>from the left</u>, x^3 is a negative numbers. So

$$-x^3 \ge x^3\cos\left(\frac{2}{x}\right) \ge x^3, \text{ or}$$

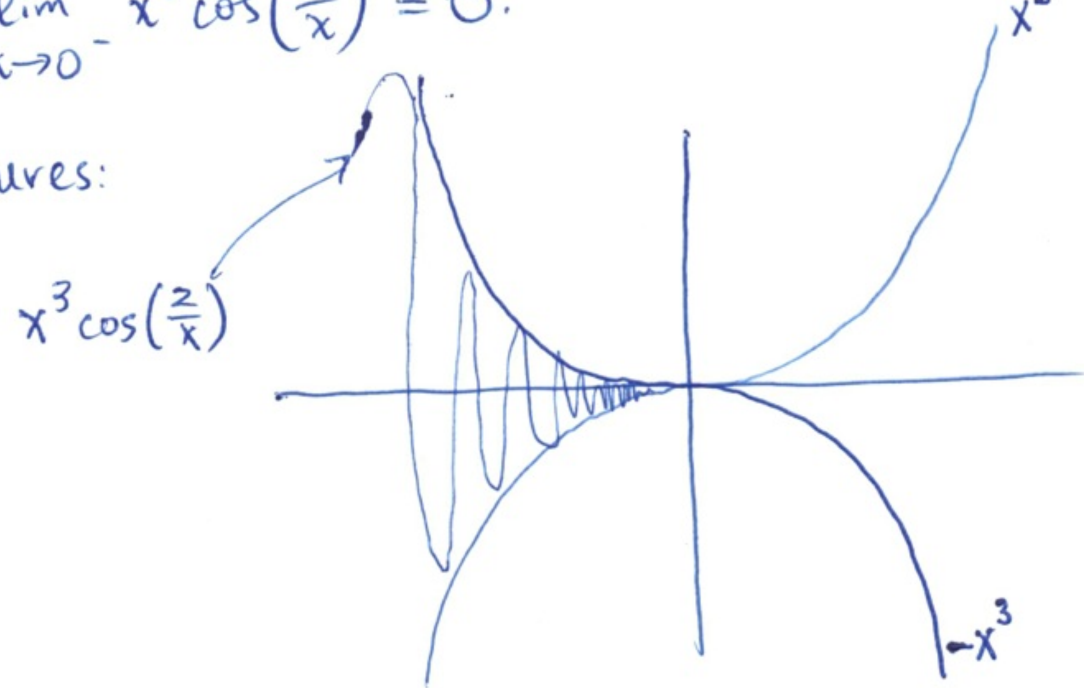
$$x^3 \le x^3\cos\left(\frac{2}{x}\right) \le -x^3$$

Now we can apply limit laws to x^3 and $-x^3$!

Since $\lim_{x\to 0^-} x^3 = 0$ and $\lim_{x\to 0^-} -x^3 = 0$,

then $\lim_{x\to 0^-} x^3\cos\left(\frac{2}{x}\right) = 0$.

In pictures:



§2.5 Continuity

A function is continuous if its graph is very nice. Formally, a function f is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$. We say "$f$ is continuous" if f is continuous for every value of a.

Rewordings:

① A function $f(x)$ is continuous at $x=a$ if the limit at a can be found by plugging in $x=a$.

② A function $f(x)$ is continuous at $x=a$ if the left and right limits both exist at $x=a$, and they're both equal to the number $f(a)$.

Terminology.

If f is not defined at $x=a$ or $\lim_{x \to a} f(x)$ doesn't exist, we say f is discontinuous.

Example: Suppose

$$f(x) = \begin{cases} \dfrac{(x+3)(x-1)}{x+3} & \text{if } x < -2 \\[2ex] \dfrac{6}{x} & \text{if } -2 \le x < 1 \\[2ex] x^2+1 & \text{if } x \ge 1. \end{cases}$$

where is x discontinuous?

Solution: First, we look for places where $f(x)$ is not defined—it's discontinuous there.

The formula for $f(x)$ gives problems at $x=-3$ and $x=0$. we get $f(-3)=\frac{0}{0}$ and $f(0)=\frac{6}{0}$. So f is discontinuous at $x=-3$ and $x=0$.

Other problem spots: $x=-2$ & $x=1$. We check: $x=-2$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{(x+3)(x-1)}{x+3} = \frac{(-2+3)(-2-1)}{-2+3} = -3.$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{6}{x} = \frac{6}{-2} = -3.$$

So $\lim_{x\to -2} f(x) = -3$, which is $f(-2)$ since $f(-2)=\frac{6}{-2}=-3$.

$\Rightarrow$ continuous @ $x=-2$.

Check $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{6}{x} = \frac{6}{1} = 6$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+1) = 1^2+1 = 2.$$

Left limit $\neq$ right limit, so $\lim_{x\to 1} f(x)$ doesn't exist.

$\Rightarrow$ f is discontinuous at $x=1$.

We say <u>f is continuous on an interval</u> if f is continuous at every point in the interval.

<u>Example</u>: Is $f(x) = \begin{cases} \frac{(x+3)(x+1)}{(x+2)} & \text{if } x \le 0 \\ 3x^2 + x + \frac{3}{2} & \text{if } x > 0 \end{cases}$

continuous on $[-1, 1]$?

<u>Solution</u>: The function f obviously has problems at $x=-2$ and $x=0$, but $x=-2$ doesn't matter since we are only asked about $[-1,1]$. We check $x=0$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{(x+3)(x+1)}{x+2} = \frac{(3)(+1)}{2} = +\frac{3}{2}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(3x^2 + x + \frac{3}{2}\right) = 3 \cdot 0^2 + 0 + \frac{3}{2} = \frac{3}{2}.$$

So $\lim_{x \to 0} f(x) = \frac{3}{2} = f(0)$, and f is continuous on $[-1,1]$.

<u>Thankfully we know</u>:

These functions are continuous at every point in their domain:

- polynomials $(x^3 + x^2 + 2x - 5)$
- rational functions $\frac{p(x)}{q(x)}$
- root functions $\sqrt[5]{x}$

- trig, inverse trig ($\cos(x)$, $\tan^{-1}(x)$)
- exponential and log/ln functions.

<u>Why introduce this new word?</u>

- Being continuous at $x=a$ is <u>nicer</u> than having a limit at $x=a$. (There's a limit, and in fact it's $f(a)$).
- Continuous functions are exactly the ones for which you can calculate limits by just plugging in numbers.
