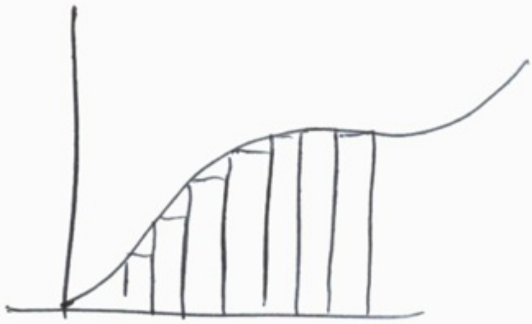
MATH 1500 Lecture 34.

Last day we saw that we can approximate the area under a curve by rectangles, ie



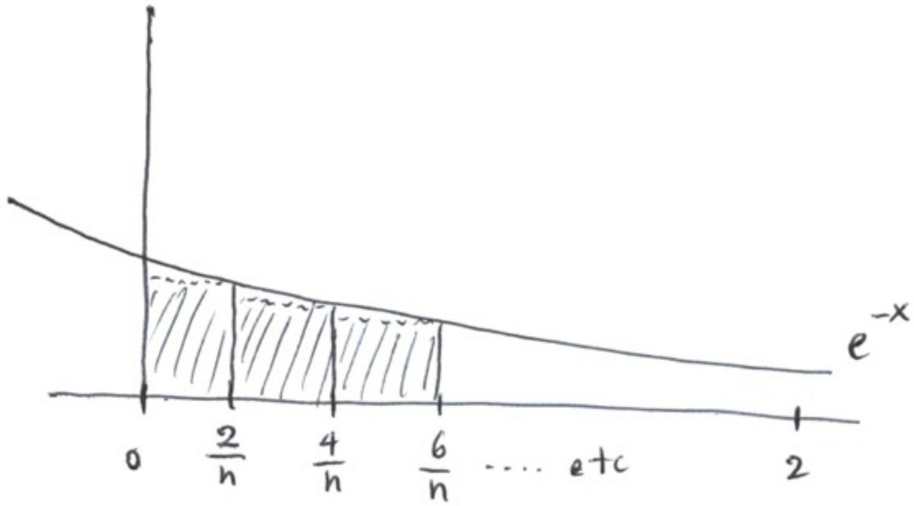
Then in order to guess the area, we can pack in thinner and thinner rectangles.

In general, if we pack 'n' rectangles under a curve and add up their areas, we get a number R_n. By packing in more rectangles, R_n becomes closer to the actual area, A. I.e.

$$A = \lim_{n \to \infty} R_n.$$

Example. Express the area under $f(x) = e^{-x}$ from $x=0$ to $x=2$ as a limit.

Solution: If we divide the interval from 0 to 2 into n equally long pieces, each piece has length $\frac{2-0}{n} = \frac{2}{n}$.



Now we can choose left or right hand endpoints to calculate the heights of the corresponding rectangles. Say we choose right hand endpoints.

Then
$$R_n = \underbrace{\frac{2}{n}e^{-2/n} + \frac{2}{n}e^{-4/n} + \frac{2}{n}e^{-6/n} + \dots + \frac{2}{n}e^{-2n/n}}_{\text{Here I am adding up } n \text{ terms.}}$$

Then
$$\text{Area} = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + \dots + e^{-2n/n}\right)$$

<u>or</u> $= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} e^{-2i/n}$ (Sigma notation).

<u>Definition</u>: Suppose $f(x)$ is a function defined on $[a,b]$. The <u>definite integral from a to b</u> is written
$$\int_a^b f(x)\,dx$$

and it is a number that is equal to the area between $f(x)$ and the x-axis from a to b, if $f(x) \geq 0$.

Therefore we must have:

area of rectangles

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

height of rectangles　　width of rectangles

<u>Example</u>: Evaluate $\int_0^4 x^2\,dx$ (I.e, find the area under $f(x)=x^2$ from $x=0$ to $x=4$).

<u>Solution</u>:

If we use n rectangles, their widths are $\frac{4}{n}$.

Right-hand endpoints gives heights of $\left(\frac{4}{n}\right)^2$, $\left(\frac{2\cdot 4}{n}\right)^2$, $\left(\frac{3\cdot 4}{n}\right)^2$, etc. So

$$R_n = \frac{4}{n}\cdot\left(\frac{4}{n}\right)^2 + \frac{4}{n}\left(\frac{2\cdot 4}{n}\right)^2 + \frac{4}{n}\left(\frac{3\cdot 4}{n}\right)^2 + \dots + \frac{4}{n}\left(\frac{n\cdot 4}{n}\right)^2$$

$$= \frac{4}{n}\cdot\left(\frac{4}{n}\right)^2\cdot\left(1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2\right)$$

$$= \frac{4^3}{n^3}\cdot\left(\frac{n(n+1)(2n+1)}{6}\right)$$

well-known formula for sum of squares.

$$= \frac{4^3}{n^3}\left(\frac{(n^2+n)(2n+1)}{6}\right) = \frac{4^3}{n^3}\left(\frac{2n^3+2n^2+n^2+n}{6}\right)$$

$$= \frac{4^3(2n^3+3n^2+n)}{6n^3}$$

So the area is $A = \int_0^4 x^2\,dx = \ldots$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{4^3}{6}\cdot\left(\frac{2n^3+3n^2+n}{n^3}\right)$$

$$= \frac{4^3}{6}\cdot\lim_{n\to\infty}\left(\frac{2n^3/n^3 + \cancel{3n^2/n^3} + \cancel{n/n^3}}{n^3/n^3}\right)$$

$$= \frac{4^3}{6}\cdot 2 = \frac{64}{3}$$ (Agrees with last day).

Note: The sum of areas of rectangles

$$R_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

is called the Riemann sum, named after Bernhard Riemann (1826–1866).

Example: Use area computations to evaluate:

(a) $\int_0^1 \sqrt{1-x^2}\,dx$, (b) $\int_0^3 (x-1)\,dx$.

Solution:

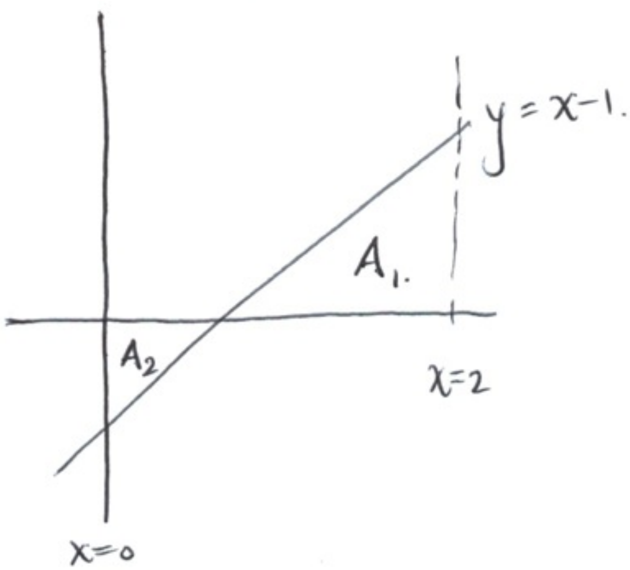
We can avoid Riemann sums by using our knowledge of areas.

a) The graph of $f(x) = \sqrt{1-x^2}$ is a quarter circle above the x-axis:

Since the area of a circle is $A = \pi r^2$, the area of a quarter circle with radius 1 is:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\left(\pi (1)^2\right) = \frac{\pi}{4}.$$

b) The line $y = x-1$ crosses the x-axis at $x=1$:



Imagine packing rectangles between the x-axis and $y = x-1$:

If the area of one of n rectangles is computed using the formula:

$$\underbrace{\frac{3}{n}}_{\Delta x} \cdot \underbrace{\left(\frac{3i}{n} - 1\right)}_{f(x_i^*)}$$

then rectangles to the left of $x=1$ come with a <u>negative</u> sign in front of their areas, since $y=x-1$ is negative there.

<u>Fact</u>: Areas <u>below</u> the x-axis are counted with a negative sign! So

$$A = \int_0^3 (x-1)\,dx = A_1 - A_2 = \frac{1}{2}(2\cdot 2) - \frac{1}{2}(1\cdot 1) = \frac{3}{2}.$$

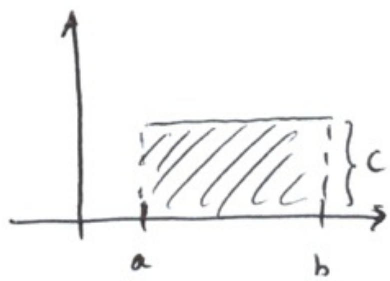
MATH 1500 April 2.

§5.2.

Last day we saw that $\int_a^b f(x)\,dx$ is equal to the area between f(x) and the x-axis, with areas below the x-axis counted negatively. This is because $\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n$, and rectangles below the x-axis have 'negative height'.

Important properties of the integral:

① For any constant c, $\int_a^b c\,dx = c(b-a)$, the area of a rectangle:



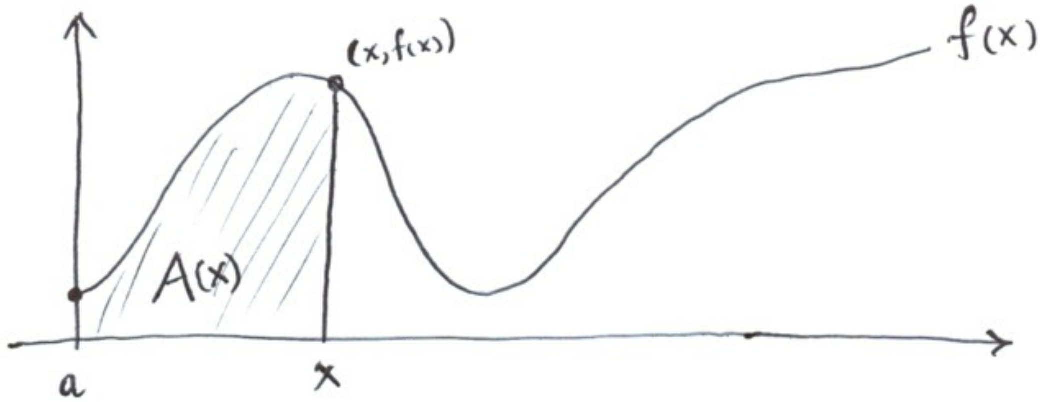
② $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (areas add vertically).

③ $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ (scaling areas)

④ $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (areas add horizontally).

The Fundamental theorem of calculus or how to actually evaluate integrals.

Imagine you choose a function $f(x)$, and it looks like:



Imagine you have a computer program written for this function that displays the following outputs:

x-value:

area $A(x)$:

Now we start increasing the x-value at a steady rate. As x increases, we see:

- $A(x)$ increases faster when $f(x)$ is high above the x-axis
- $A(x)$ increases slowly when $f(x)$ is smaller.

In other words, the rate of change of the area is proportional to $f(x)$. In fact,

$$\frac{dA}{dx} = f(x)$$

But the notation for area introduced last day is

$$A(x) = \int_a^x f(t)\,dt$$

so the fundamental theorem of calculus is:

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = \frac{dA}{dx} = f(x).}$$

(This is true as long as $f(x)$ is continuous)

This essentially says that by differentiating the area function $A(x)$, we get $f(x)$. So, if we're given the function $f(x)$ and want to know the area function $A(x)$, we must anti-differentiate.

Fundamental theorem of calculus, part II:

Suppose $f(x)$ is continuous and $F(x)$ is an antiderivative of $f(x)$. Then since $F(x)$ is an 'area function' for $f(x)$,

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

So this method (taking antiderivatives) replaces "Riemann sums", the definition.

Example: What is the area under $f(x) = x^2$ from $x=0$ to $x=4$?

Solution: The area is

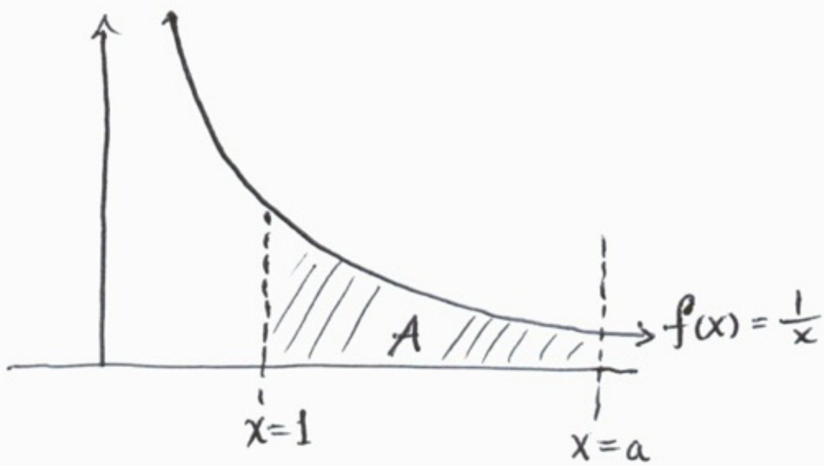
$$\int_0^4 x^2\,dx.$$ An antiderivative of x^2 is $\frac{x^3}{3}$.

By the fundamental theorem of calculus

$$\int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{4^3}{3} - \frac{0^3}{3} = \frac{4^3}{3} = \frac{64}{3}.$$

this notation means to plug in the top and bottom numbers then subtract.

Example: What number 'a' makes it so that the area A under the curve below is equal to 1?



Solution: We want a so that

$$\int_1^a \frac{1}{x}\,dx = 1.$$

An antiderivative of $\frac{1}{x}$ is $\ln(x)$. By the fundamental theorem of calculus,

$$1 = \int_1^a \frac{1}{x}\,dx = \Big[\ln(x)\Big]_1^a = \ln(a) - \ln(1) = \ln(a) - 0$$
$$= \ln(a).$$

So the number 'a' must satisfy $\ln(a) = 1$, i.e. we must have $a = e$.

Example: Evaluate $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz$.

Solution: By the rules of itegrals,

$$\int_1^{18} \sqrt{\frac{3}{z}}\,dz = \int_1^{18} \sqrt{3}\cdot\frac{1}{\sqrt{z}}\,dz$$

$$= \sqrt{3}\int_1^{18} (z)^{-1/2}\,dz$$

$$= \sqrt{3}\ \cancel{\int_1^{18} \frac{z^{-1/2+1}}{-1/2+1}\,dz}\ \left[\frac{z^{1/2}}{-1/2+1}\right]_1^{18} \quad \text{(FTC)}.$$

$$= \sqrt{3}\,\Big[2\sqrt{z}\Big]_1^{18}$$

$$= \sqrt{3}\,\big(2\sqrt{18} - 2\sqrt{1}\big)$$

$$= 2\sqrt{3}\,\big(\sqrt{18} - 1\big).$$

Example: What is the derivative of the function

$g(x) = \int_3^x e^{t^2-t}\,dt$. ?

Solution:

$$\frac{dg}{dx} = \frac{d}{dx}\int_3^x e^{t^2-t}\,dt$$

$$= e^{x^2-x} \qquad \text{(really, it's that easy).}$$

Example: What is the derivative of

$$y = \int_0^{x^4} \cos\theta \, d\theta \ ?$$

Solution: Note that y is a composition of two functions whose derivatives we know:

$$f(u) = \int_0^u \cos\theta \, d\theta$$

and $u(x) = x^4$, then $y = f(u(x))$.

So $\frac{dy}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$. Then $\frac{df}{du} = \cos(u)$ (FTC),

and $\frac{du}{dx} = 4x^3$. Overall,

$$\frac{dy}{dx} = \cos(u)\cdot 4x^3 = \cos(x^4)\cdot 4x^3.$$

MATH 1500 April 4th.

§5.3 Questions 1-48, 55-63.

Last day we saw the fundamental theorem of calculus, which comes in two parts:

Part I: If f is continuous on $[a,b]$ then the function

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ too, and $\frac{dg}{dx} = f(x)$.

(Think of this like a new derivative rule).

Part II: If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(This is a rule for calculating areas).

Example: Find the derivative of the function

$$g(x) = \int_x^0 \sin\left(\frac{1+t}{\sqrt{t}+1}\right) dt.$$

Solution: This is almost a direct application of the fundamental theorem of calculus. However, to apply the theorem we need a number at the bottom of the integral sign, and a variable at the top.

So there is a new rule:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

[By FTC II, if $f(x)$ has an antiderivative $F(x)$ then all we're saying is $F(b)-F(a) = -(F(a)-F(b))$].

So

$$g(x) = \int_x^0 \sin\left(\frac{1+t}{\sqrt{t}+1}\right)dt = -\int_0^x \sin\left(\frac{1+t}{\sqrt{t}+1}\right)dt$$

Then $\dfrac{dg}{dx} = -\dfrac{d}{dx}\left(\int_0^x \sin\left(\frac{1+t}{\sqrt{t}+1}\right)dt\right) = -\sin\left(\frac{1+x}{\sqrt{x}+1}\right)$

by the fundamental theorem of calculus.

<u>Example</u>: What is the derivative of

$$g(x) = \int_{-e^x}^{e^x} t\cos(t)\,dt\,?$$

<u>Solution</u>: We have to use integration tricks to re-write $g(x)$ as integrals $\int_c^{h(x)}$, where the bottom is a constant c and $h(x)$ is some function of x. This brings us closer to applying the FTC.

The trick: Recall that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \left(\begin{array}{c}\text{areas add}\\ \text{horizontally}\end{array}\right)$$

So

$$\int_{e^{-x}}^{e^x} t\cos(t)\,dt = \int_{e^{-x}}^{0} t\cos t\,dt + \int_0^{e^x} t\cos t\,dt$$

$$= -\int_0^{e^{-x}} t\cos t\,dt + \int_0^{e^x} t\cos t\,dt.$$

Now we differentiate each piece using FTC, and chain rule.

Here's one piece:

if $-\int_0^{e^{-x}} t\cos t\,dt$, then set $f(u) = -\int_0^{u} t\cos t\,dt$,

$u(x) = e^{-x}$. Then the piece is

$$-\int_0^{e^{-x}} t\cos t\,dt = f(u(x)) \quad \text{so its derivative is}$$

$$\frac{df}{du}\cdot\frac{du}{dx} = \underbrace{u\cos u}_{\text{FTC}}\cdot(-1)e^{-x} = e^{-x}\cos(e^{-x})\cdot(e^{-x})\,..$$

Similarly, the derivative of the other piece is

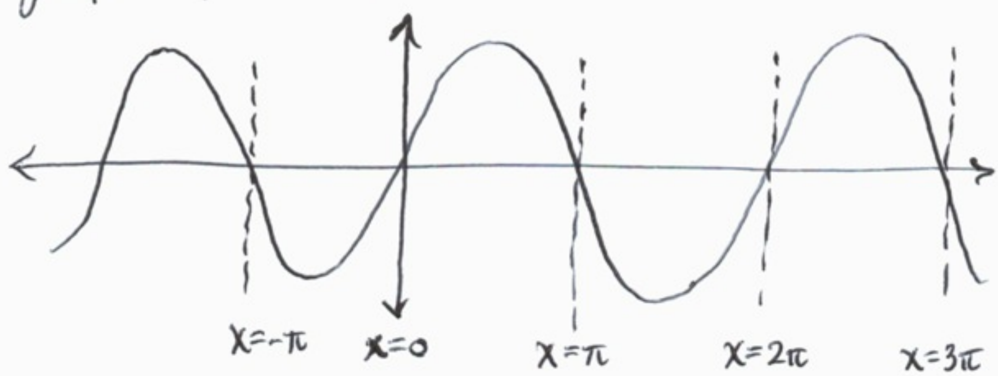
$e^{x}\cos(e^{x})\cdot e^{x}$, overall

$$\frac{dg}{dx} = \underbrace{-e^{-2x}\cos(e^{-x})}_{\text{1st piece}} + \underbrace{e^{2x}\cos(e^{x})}_{\text{second piece}}.$$

Example: What is the total area between $f(x) = \sin(x)$ and the x-axis, from $x = -\pi$ to $x = 5\pi$?

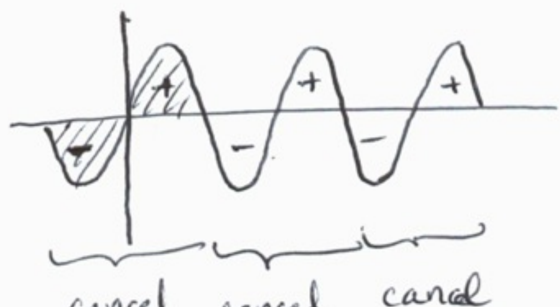
Solution:

The graph of $\sin(x)$ is



Remembering that areas below the x-axis are counted negatively, if we do

$\int_{-\pi}^{5\pi} \sin(x)\,dx$ then we get



$$= \Big[-\cos(x)\Big]_{-\pi}^{5\pi} = -\cos(5\pi) - (-\cos(-\pi))$$

$$= -(-1) + (-1)$$

$$= 0, \text{ because all the area cancels}$$

The correct answer is:

$$\text{Area} = 6\int_0^{\pi} \sin x\,dx = 6\Big[-\cos(x)\Big]_0^{\pi} = 6\big(-\overset{-1}{\cos(\pi)} - (-\overset{1}{\cos(0)})\big)$$

$$= 6(1+1) = 12.$$

I.e. The area under one hump is 2, there are 6 humps.

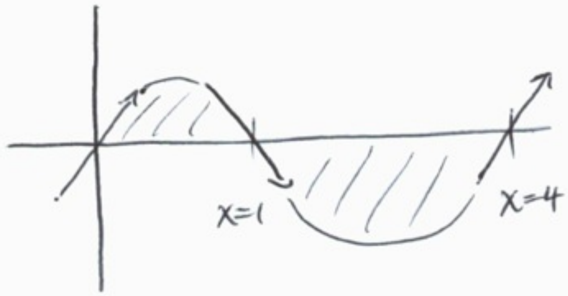
<u>Example</u> Find the total area between

$y = x^3 - 5x^2 + 4x$ and the x-axis.

<u>Solution</u>: It factors as

$y = x(x-4)(x-1)$, so it crosses the x-axis at

$x = 0, 1, 4$. By testing concave up/concave down we find



So the total area will be

$$\int_0^1 x^3 - 5x^2 + 4x\,dx - \int_1^4 x^3 - 5x^2 + 4x\,dx$$

fixes the negative area problem.

$$= \left[\frac{x^4}{4} - \frac{5}{3}x^3 + \frac{4x^2}{2}\right]_0^1 - \left[\frac{x^4}{4} - \frac{5}{3}x^3 + 2x^2\right]_1^4$$

$$= \left(\frac{1}{4} - \frac{5}{3} + 2\right) - (0 - 0 + 0) - \left[\left(4^3 - \frac{5}{3}\cdot 4^3 + 2\cdot 4^2\right) - \left(\frac{1}{4} - \frac{5}{3} + 2\right)\right]$$

$$= \frac{7}{12} - \left(-\frac{45}{4}\right) = \frac{71}{6}$$ is the total area.