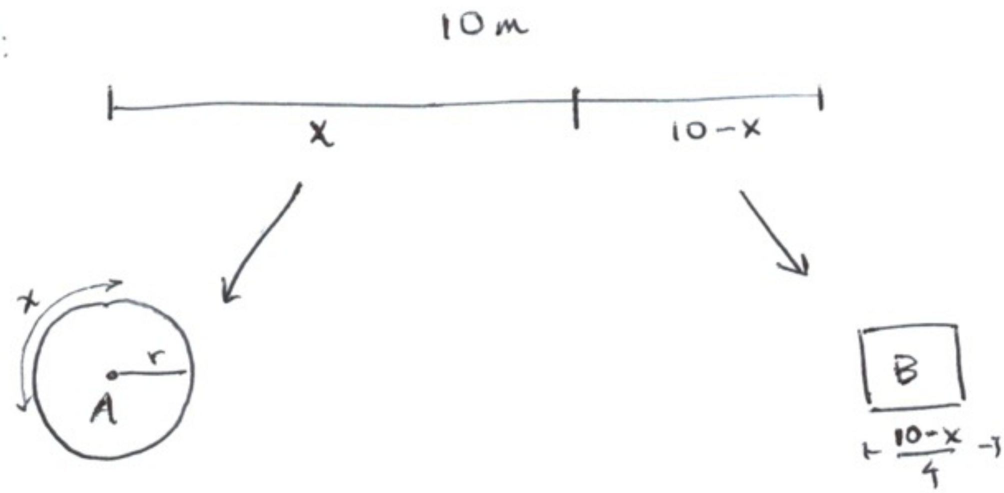
MATH 1500 March 24.

Example: A wire 10m long is cut into two pieces, one piece is bent into a circle and the other into a square. Where should the wire be cut to minimize the total area of the two figures? To maximize the area?

Solution:



The area of the circle is $A = \pi r^2$, and r is related to the circumference x by $x = 2\pi r$

$$\Rightarrow r = \frac{x}{2\pi}.$$

So $$A = \pi \left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$$

The area of the square is $B = \left(\frac{10-x}{4}\right)^2$, so the total area is

$$T = A + B = \frac{x^2}{4\pi} + \left(\frac{10-x}{4}\right)^2.$$ Note that since the wire is only 10 m long, $0 \le x \le 10$, with $x = 0$ meaning we use all of it for a square, $x = 10$ means we use it all for a circle.

So where is the absolute min and max for $T(x)$ with $0 \le x \le 10$?

Calculate $T'(x) = \frac{2x}{4\pi} + (2)\left(\frac{10-x}{4}\right)\left(\frac{-1}{4}\right)$

$$= \frac{x}{2\pi} + \left(\frac{-1}{2}\right)\left(\frac{10-x}{4}\right)$$

$$= \frac{x}{2\pi} - \frac{(10-x)}{8}$$

$$= \frac{8x - 2\pi(10-x)}{16\pi}$$

So $T'(x) = 0$ gives $8x - 20\pi + 2\pi x = 0$

or $(8+2\pi)x = 20\pi$

$$x = \frac{20\pi}{8+2\pi} \simeq 4.4.$$

This is a local min since $T''(x) = \frac{1}{2\pi} + \frac{1}{8} > 0$, so $T(x)$ is concave up there. We need only compare it to $T(0) = 0 + \left(\frac{10-0}{4}\right)^2 = \frac{100}{16} \simeq 6.25$

and $T(10) = \frac{10^2}{4\pi} \simeq 7.96$.

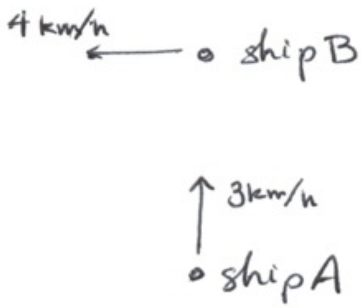
So to get the largest area ($\simeq 7.96 m^2$) we make one big circle. To get the smallest area we cut at $x = \frac{20\pi}{8+2\pi}$ m and make a circle and square.

(Total area in this case is $\simeq 3.5$, we already knew it was a min from the second derivative test).

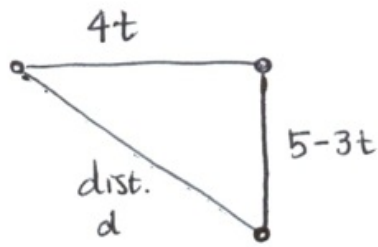
Example (2007 final)

At 1pm, ship A is 5km south of ship B. Ship A sails north at 3km/h, and ship B sails west at 4km/h. When are the ships closest, and how far apart are they at that time?

Solution:



So



So we want to minimize $d(t)$, and it obeys the eqn:

$$(d(t))^2 = (4t)^2 + (5-3t)^2$$

$$\Rightarrow d(t) = \sqrt{(4t)^2 + (5-3t)^2}.$$

Now for the purpose of finding t that makes $d(t)$ minimum, it's okay to ignore the square root <u>since $f(x) = \sqrt{x}$ is an increasing function</u>.

However, let's keep the square root for now since we already saw an example where we ignored it.

Then

$$d'(t) = \frac{1}{2}\,\frac{1}{\sqrt{(4t)^2 + (5-3t)^2}} \cdot \left(2\cdot 16t + 2(5-3t)(-3)\right)$$

$$= \frac{32t - 30 + 18t}{2\sqrt{(4t)^2 + (5-3t)^2}} = \frac{50t - 30}{2\sqrt{(4t)^2 + (5-3t)^2}}$$

So then $d'(t)=0$ gives $50t-30=0$

$$\Rightarrow t=\frac{3}{5} \text{ hours.}$$

Since $d''(t)$ is hard to compute, we can check it is a min using the first derivative test:

If $t<\frac{3}{5}=\frac{30}{50}$ then $50t<30$

$$\Rightarrow 50t-30<0$$

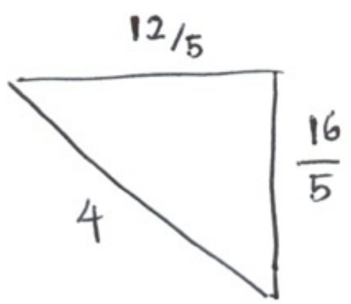
and so $t>\frac{3}{5} \Rightarrow 50t-30>0$.

So $d'(t)=\dfrac{50t-30}{2\sqrt{(\text{stuff})}}$ is negative to the left of $\frac{3}{5}$, pos to the right. Thus $d(t)$ is decreasing to the left of $3/5$, incr. to the right. So $t=\frac{3}{5}$ is a min, or in other words

1pm + $60\cdot\frac{3}{5}$ min $=$ 1:36 pm.

At that time the distance is $d(\frac{3}{5})=\sqrt{4(\frac{9}{25})+\text{etc}\cdots}$

$$\simeq 4.$$



<u>Example</u>: If the ~~difference~~ sum of two numbers is 9, maximize the product of their square roots.

<u>Solution</u>: No picture, but call the two numbers x and y.

Then $x+y=9$ and maximize

$$P=\sqrt{x}\cdot\sqrt{y}=\sqrt{xy}$$

Then $y = 9 - x$ so P is a function of x with

$$P(x) = \sqrt{x(9-x)} = \sqrt{9x - x^2}.$$

Then $P'(x) = \dfrac{9-2x}{2\sqrt{9x-x^2}}$, so $P'(x) = 0$ gives

$9 - 2x = 0$.

$\Rightarrow x = \frac{9}{2}$. Correspondingly $y = \frac{9}{2}$ as well.

Is this a max? We find $9 - 2x > 0$ left of $x = 9/2$, and $9 - 2x < 0$ right of $x = 9/2$. So $P(x)$ is increasing on $(-\infty, 9/2)$ and decreasing on $(9/2, \infty)$ and $x = 9/2$ is a max.

The two numbers are $x = 9/2$, $y = 9/2$.

MATH 1500 Calculus March 26

§4.9 1-17, 20-22, 25-43, 45-52, 59-65. **Armes 208**

This stuff is important! April 14 9:00-12:00

We learned how to take derivatives:

$$f(x) \xrightarrow{\text{differentiate}} f'(x), \text{ a derivative.}$$

Nowe we learn how to go backwards

$$f(x) \xleftarrow{\text{integrate}} f'(x)$$

an antiderivative.

<u>Definition</u>: A function $F(x)$ is called an <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$.

<u>Remark</u>: If $F'(x) = f(x)$, then $(F(x)+C)' = F'(x)+(C)'$

$$= f(x)+0 = f(x),$$

since the derivative of a constant is 0. So in general, if $F(x)$ is an antiderivative, so is $F(x)+C$ for any constant C. So, we think of $F(x)+C$ as "the most general antiderivative" of $f(x)$.

<u>Example</u>: What is the most general antiderivative of $f(x) = x^n$?

<u>Solution</u>: We can solve this question by answering: What function would you have to differentiate to get

x^n? Since

$$\frac{d}{dx}(cx^k) = ck\,x^{k-1}$$

want this to equal the derivative of something

This means we would need $x^n = ck\,x^{k-1}$

This is the derivative of something.

So the exponents give

$$n = k-1 \quad \text{or} \quad k = n+1$$

and the coefficients give

$$ck = 1$$

$$\text{or} \quad c = \frac{1}{k} = \frac{1}{n+1}.$$

So we know that $\dfrac{d}{dx}\left(\dfrac{1}{n+1}x^{n+1}\right) = x^n$. Therefore

the most general antiderivative of x^n is $\dfrac{1}{n+1}x^{n+1} + C$.

In general, for every derivative rule we learned there is an opposite "antiderivative rule"; i.e. an integration rule.

Here is a list of relevant antiderivative rules: (Here, $F'(x) = f(x)$ and $G'(x) = g(x)$)

Function	Particular antiderivative	Corresponding derivative rule.
$cf(x)$	$cF(x)$	$\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$
$f(x)+g(x)$	$F(x)+G(x)$	$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx} + \frac{dg}{dx}$

$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{d}{dx}(x^n) = n x^{n-1} \ (n \neq 0)$.
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{d}{dx} \ln\lvert x\rvert = \frac{1}{x}$.
e^x	e^x	$\frac{d}{dx}(e^x) = e^x$
$\cos x$	$\sin x$	$\frac{d}{dx}(\sin x) = \cos x$
$\sin x$	$-\cos x$	$\frac{d}{dx}(\cos x) = -\sin x$.

Of course there are many, many, many others.

Example: Find the most general antiderivative of

$$f(x) = 3\sin x - \frac{\sqrt{x} + x^5}{4}$$

Solution: We write $f(x)$ as a sum of functions appearing in the table:

$$f(x) = 3\sin x - \frac{\sqrt{x}}{4} - \frac{x^5}{4}$$

$$= 3\sin x - \frac{1}{4}x^{1/2} - \frac{1}{4}x^5$$

So the general antiderivative is:

$$F(x) = 3(-\cos x) - \frac{1}{4}\left(\frac{x^{1/2+1}}{1/2+1}\right) - \frac{1}{4}\,\frac{x^{5+1}}{5+1} + C$$

$$= -3\cos x - \frac{1}{4}\left(\frac{x^{3/2}}{3/2}\right) - \frac{1}{24}x^6 + C$$

$$= -3\cos x - \frac{1}{6}x^{3/2} - \frac{1}{24}x^6 + C$$

DONE.

In general, 'C' remains an unknown variable. However, there are times when you will be given some information that you use to solve for C.

<u>Example</u>: If $f'(x) = e^x - \cos(x) + x^2$ and $f(0) = 3$, then what is $f(x)$?

<u>Solution</u>: The most general possible antiderivative of $f'(x) = e^x - \cos x + x^2$ is

$$f(x) = e^x - \sin x + \frac{x^3}{3} + C$$

But we want a particular antiderivative with $f(0) = 3$, in other words

$$3 = f(0) = e^0 - \sin 0 + \frac{0^3}{3} + C$$

$$\Rightarrow \quad 3 = 1 - 0 + 0 + C$$

$$\Rightarrow C = 3 - 1 = 2.$$

So the antiderivative we're looking for is $f(x) = e^x - \sin x + \frac{x^3}{3} + 2$

The equation "$f'(x) = $ stuff", or in general any equation with derivatives in it, is called a differential equation. If $f'(x)$ appears in a differential equation, <u>solving</u> the equation means determining the antiderivative $f(x)$. The extra condition $f(a) = b$ lets you determine C exactly in your antiderivative.

Example: Find $f(x)$ if $f''(x) = 2x^2 - 1$, $f'(0) = 3$ and $f(0) = 1$.

Solution: From $f''(x) = 2x^2 - 1$ we get

$$f'(x) = 2\left(\frac{1}{3}x^3\right) - x + C = \frac{2}{3}x^3 - x + C$$

So $f'(0) = 3$ gives

$$3 = f'(0) = \frac{2}{3} \cdot 0 - 0 + C \Rightarrow C = 3.$$

So $f'(x) = \frac{2}{3}x^3 - x + 3$. Then

$$f(x) = \frac{2}{3}\left(\frac{1}{4}x^4\right) - \frac{1}{2}x^2 + 3x + D,$$ and $f(0) = 1$ gives

$$1 = f(0) = \frac{1}{6}0^4 - \frac{1}{2}0^2 + 3 \cdot 0 + D$$

$$\Rightarrow D = 1$$

So $f(x) = \frac{1}{6}x^4 - \frac{1}{2}x^2 + 3x + 1.$

MATH 1500 Friday, March 28.

§ 4.9, 5.1.

Last day we saw antiderivatives. If we're given a function $f(x)$, an antiderivative of f is a function $F(x)$ with $F'(x) = f(x)$.

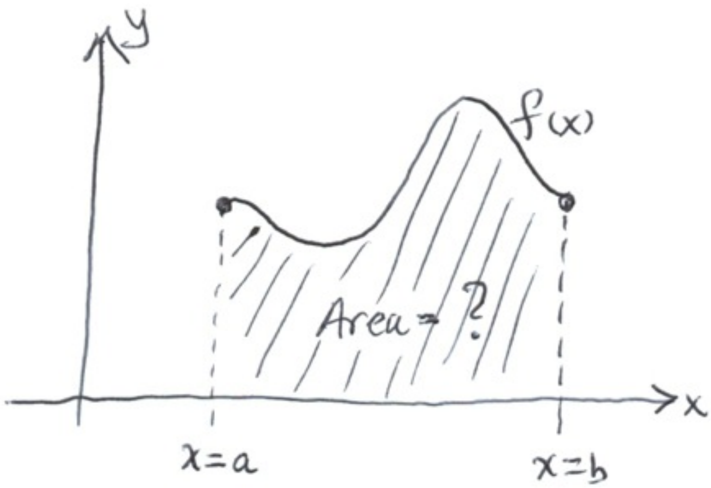
For example, $\frac{x^4}{4}$ is an antiderivative of x^3 since

$$\frac{d}{dx}\left(\frac{1}{4}x^4\right) = \frac{1}{4}\cdot 4x^3 = x^3.$$

Antiderivatives are used when solving differential equations, but also in calculating areas.

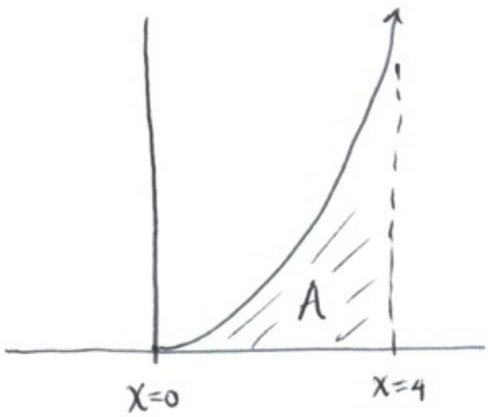
We study the problem of calculating areas today and show how antiderivatives are used to solve the problem during the classes that follow.

Problem: Given a function $f(x)$ and an interval $[a,b]$, what is the area between the graph of $f(x)$ and the x-axis from $x=a$ to $x=b$?



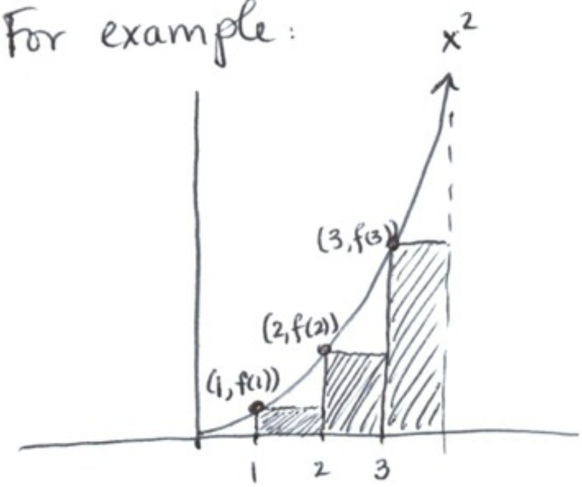
Example: Estimate the area under $y = x^2$ from $x = 0$ to to $x = 4$.

Solution: The graph looks like:



We can guess the area A by packing it with rectangles.

For example:

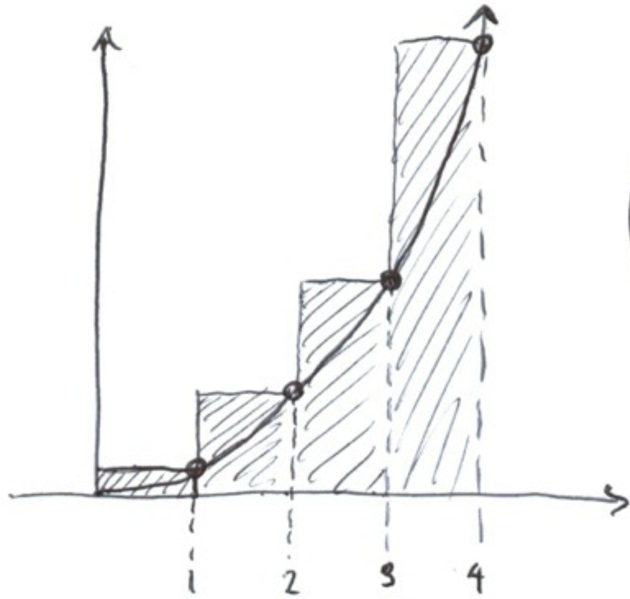


(left endpoints) of rectangles

So the area of the rectangles is

$$1 \times f(1) + 1 \times f(2) + 1 \times f(3) = 1^2 + 2^2 + 3^2$$
$$= 1 + 4 + 9 = 14.$$

So, $A > 14$. We can also make an upper limit for A by using the rectangles



(right endpoints of ~~reg~~ rectangles)

The area of the rectangles is

$$1 \times f(1) + 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$$
$$= 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$$

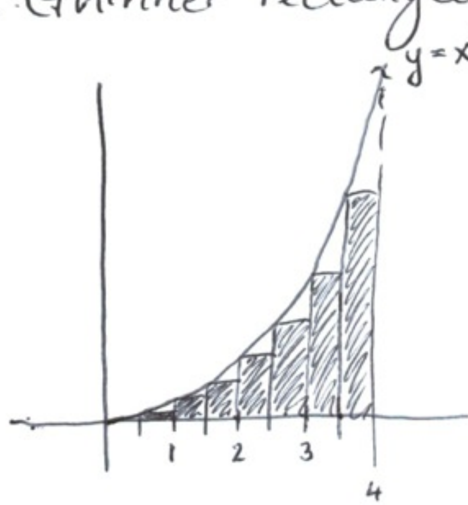
So $A < 30$. Overall, $14 < A < 30$, so we have a very rough estimate.

How can we improve our estimate (but still use rectangles, to keep it easy)?

Ans: Use thinner rectangles.

Example continued (thinner rectangles).

Suppose we do:



Then the rectangles have area

$$\frac{1}{2}\cdot f\left(\frac{1}{2}\right) + \frac{1}{2}\cdot f\left(\frac{2}{2}\right) + \frac{1}{2}\cdot f\left(\frac{3}{2}\right) + \frac{1}{2}\cdot f\left(\frac{4}{2}\right) + \cdots + \frac{1}{2} f\left(\frac{7}{2}\right)$$

$$= \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f\left(\frac{2}{2}\right) + f\left(\frac{3}{2}\right) + \cdots + f\left(\frac{7}{2}\right)\right)$$

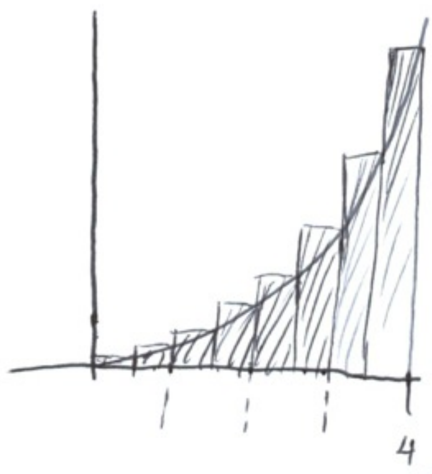
$$= \frac{1}{2}\left(\frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \cdots + \frac{49}{4}\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}\left(\sum_{i=1}^{7} i^2\right)\right)$$

the formula is $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{1}{8}\left(\frac{7(8)(15)}{6}\right) = 17.5.$$

So $A > 17.5$, similarly we can do



and find

$$\frac{1}{2}\cdot\sum_{i=1}^{8} f\left(\frac{i}{2}\right) = \frac{1}{2}\sum_{i=1}^{8}\frac{i^2}{4} = \frac{1}{8}\sum_{i=1}^{8} i^2 = \frac{1}{8}\left(\frac{8(9)(17)}{6}\right) = 25.5.$$

So $A < 25.5.$

Using smaller and smaller rectangles and by squeezing the area A between closer and closer numbers, we eventually get $A = \frac{64}{3}$.

Fact: Suppose we have chosen a function $f(x)$ and an interval $[a,b]$. Then suppose we fill under the curve with rectangles of even width, using either left or right hand endpoints. Then the area of the rectangles is: (if n rectangles)

$$R_n = \left(\frac{b-a}{n}\right) \cdot \left[f(a) + f\left(a + \frac{1(b-a)}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \dots + f\left(a + \frac{n(b-a)}{n}\right)\right].$$

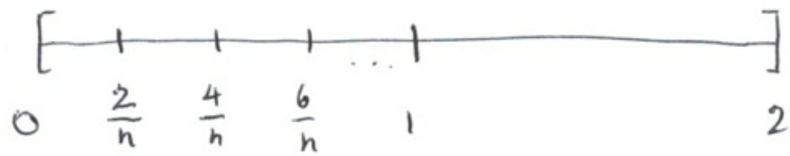
and the area is the limit of the approximations:

$$A = \lim_{n \to \infty} R_n.$$

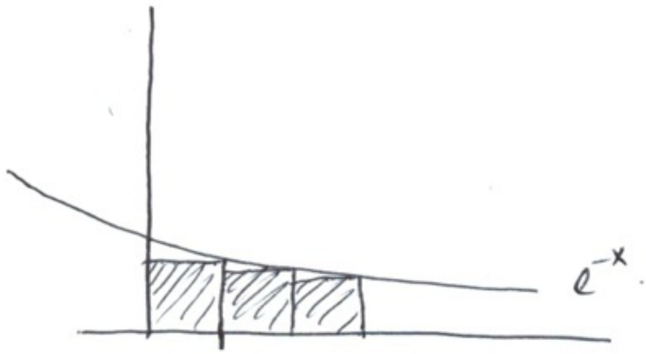
Example: Express the area under the curve $f(x) = e^{-x}$ from $x=0$ to $x=2$ as a limit.

Solution: If we divide the interval from 0 to 2 into n equally long pieces, each piece has length:

$$\frac{2-0}{n} = \frac{2}{n}.$$



Now we can use left or right hand endpoints to calculate rectangle height. Say we choose right hand endpoints:



Then $R_n = \frac{2}{n}e^{-2/n} + \frac{2}{n}e^{-4/n} + \dots + \frac{2}{n}e^{-2n/n}$

~~Area~~

$$\underbrace{\frac{2}{n}e^{-2/n} + \frac{2}{n}e^{-4/n} + \dots + \frac{2}{n}e^{-2n/n}}_{\text{here I am adding up } n \text{ terms.}}$$

Then $$\text{Area} = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + \dots + e^{-2n/n}\right).$$

$$\text{or } = \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} e^{-2i/n}$$