MATH 1500 March 17.

§ 4.5 Curve sketching. Questions 1–40

Sketching a curve is a multi-step process. We saw some of the most important steps last day:

- Finding where $f(x)$ is increasing, decreasing
- Finding where $f(x)$ is concave up/down
- Using first or second derivatives to identify local maxes and mins.

In General:

A) what is the domain of $f(x)$?

B) If it's easy to solve, solve $f(x)=0$ to get x-intercepts. what is the y-intercept $f(0)$?

C) Is the function even? Odd? Does it repeat like $\sin(x)$?

D) Are there horizontal or vertical asymptotes? (Take $\lim_{x\to\pm\infty} f(x)$ here).

E) Where is it inc/decr

F) Find max/min.

G) Where is it concave up/down?

H) MAKE THE SKETCH!

Example: Sketch $f(x) = e^{1/x}$.

Solution:

A) The domain of $f(x)$ is $(-\infty, 0) \cup (0, \infty)$.

B) Since $f(0)$ is not defined it has no y-intercept.
Since $e^{1/x} = 0$ is not possible, it also has no x-intercepts.

C) Since neither of $f(-x) = -f(x)$ or $f(-x) = f(x)$

$$e^{-1/x} \neq -e^{-1/x} \qquad e^{-1/x} \neq e^{1/x}$$

is true, the function is not even or odd.

D) Test for asymptotes?

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1,$$ so there's a horizontal asymptote at $y = 1$.

Testing the behaviour at $x=0$, we get:

Since $t = 1/x \to +\infty$ as $x \to 0^+$,

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty, \text{ and}$$

since $t = 1/x \to -\infty$ as $x \to 0^-$,

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

So $x = 0$ is a vertical asymptote.

E) To find where it is incr/decr. we calculate

$$f'(x) = e^{1/x} \cdot \frac{-1}{x^2} = -\frac{e^{1/x}}{x^2}.$$

The only critical value is $x=0$ where $f'(x)$ is not defined. We don't need a table because we can see that since $x^2>0$ and $e^{1/x}>0$ whenever $x \neq 0$, $f'(x) < 0$ whenever $x \neq 0$. Therefore $f(x)$ is decreasing everywhere.

F) $f(x)$ has no maxes or mins.

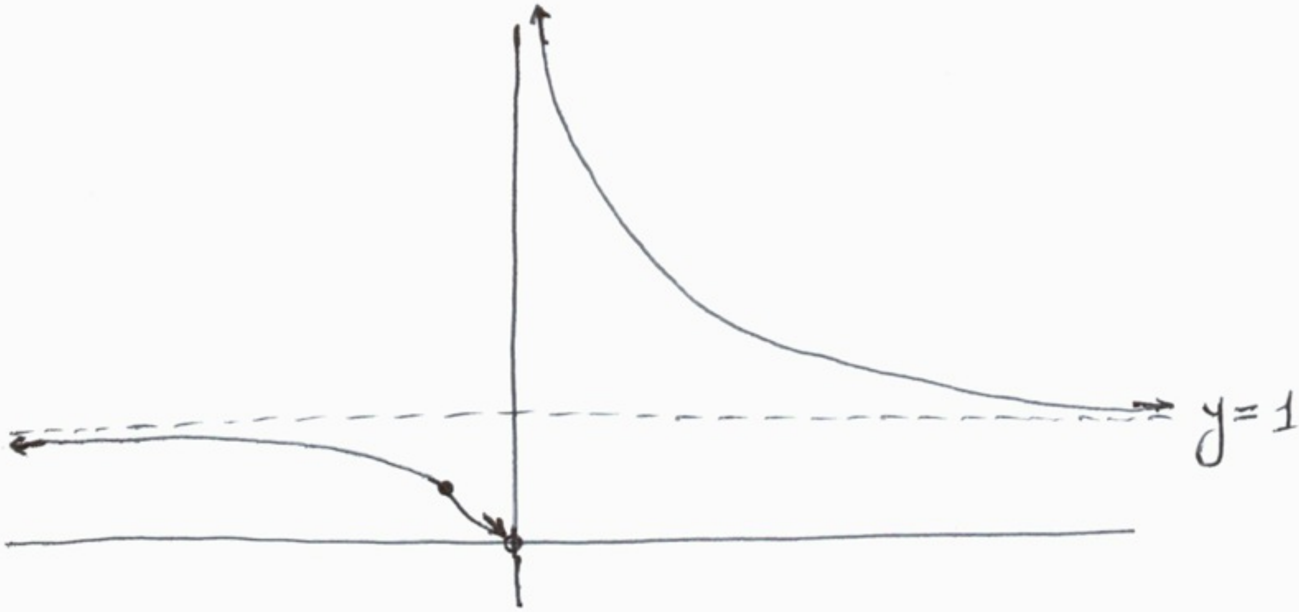
G) The second derivative is

$$f''(x) = \frac{e^{1/x}(2x+1)}{x^4}$$ (quotient rule simplifies to this).

Let's skip a table here too: $e^{1/x}$ and x^4 are positive when $x \neq 0$, and $2x+1>0$ if $x > -\frac{1}{2}$.
So $f''(x) < 0$ if $x < -\frac{1}{2}$, and $f''(x) > 0$ if $x > -\frac{1}{2}$.

So $f(x)$ is concave down on $(-\infty, -\frac{1}{2})$ and concave up on $(-\frac{1}{2}, \infty)$. The point $(-\frac{1}{2}, e^{-2})$ is an inflection point.

H) Now SKETCH!



- First draw asymptotes and small arrows to indicate limiting behaviours.
- Draw intercepts, maxes, mins and inflection points.
- Sketch, using appropriate concavity.

Example: Sketch $f(x) = \dfrac{x^2}{1-x^2}$.

Solution:

A) The domain is $\cancel{(-\infty,0)\cup(0,\infty)}$ $x \neq \pm 1$

B) The y-intercept is $f(0) = \dfrac{0^2}{1-0^2} = 0$, the x-intercept is $\dfrac{x^2}{1-x^2} = 0$

$\Rightarrow x^2 = 0 \Rightarrow x = 0.$

C) The function is even, since $(-x)^2 = x^2$ we get:

$$f(-x) = f(x)$$

D) Since $\displaystyle\lim_{x\to\pm\infty} \frac{x^2}{1-x^2} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^2}(x^2)}{\frac{1}{x^2}(1-x^2)} = \lim_{x\to\pm\infty} \frac{1}{\frac{1}{x^2}-1} = -1$

The function has a horizontal asymptote at $y=-1$.

Since $x=\pm 1$ makes the bottom zero, we get vertical asymptotes there. We test:

$\lim_{x\to -1^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = +\infty$, $\lim_{x\to 1^-} f(x) = +\infty$, $\lim_{x\to 1^+} f(x) = -\infty$.

E) We calculate $f'(x) = \dfrac{2x}{(1-x^2)^2}$. So critical values are

$x = 0, \pm 1$. Since $(1-x^2)^2$ is always positive, $f'(x)$ only changes sign when $2x$ changes sign at $x=0$.

So $f'(x) < 0$ if $x<0$ and $f'(x) > 0$ if $x>0$.

$\Rightarrow f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0,\infty)$.

F) Therefore the point $(0,0)$ is a local min.

G) We find $f''(x) = \dfrac{2+6x^2}{(1-x^2)^3}$, after some cancellation.

Since $f''(x)$ is never zero, it can only change sign at ± 1 (where it is undefined).

Skipping the table, we find

$f''(x) < 0$ on $(-\infty,-1)$, so $f(x)$ concave down

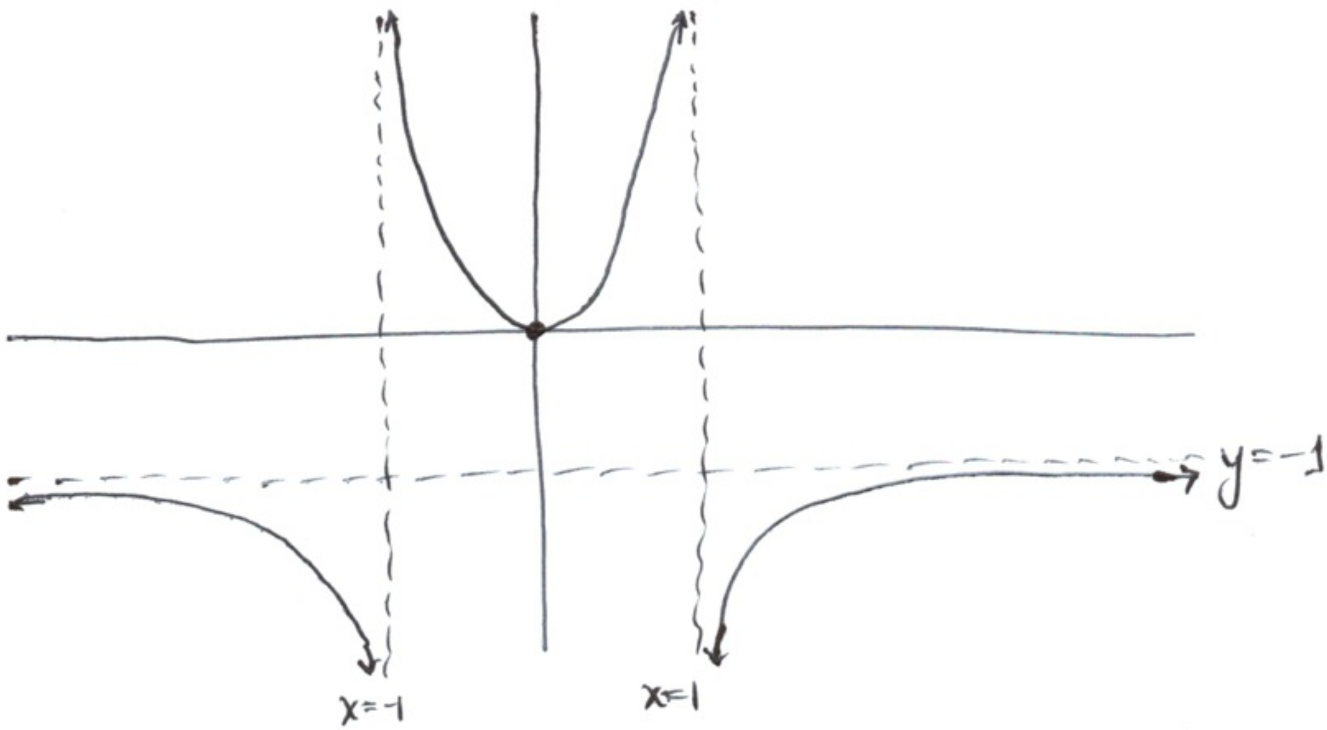
$f''(x) > 0$ on $(-1,1)$, $\Rightarrow$ concave up

$f''(x) < 0$ on $(1,\infty)$, $\Rightarrow$ concave down.

#) There are no inflection points since $f(x)$ is not defined at $x = \pm 1$ where $f''(x)$ changes sign.

SKETCH IT:

- Draw asymptotes + small arrows for limits.
- label intercepts, maxes/mins, inflection pts.
- connect the dots.



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Lecture 29.

§4.5 continued

Remark: Do not cover the material "slant asymptotes".

Example: Sketch $y = x^{5/3} - 5x^{2/3}$.

Solution:

A) Here, note that $x^{5/3} = \sqrt[3]{x^5}$ and $x^{2/3} = \sqrt[3]{x^2}$.
There are no problems with taking cube roots of negatives, so the domain is all of $\mathbb{R}$.

B) Intercepts.

The y-intercept is
$$y(0) = 0^{5/3} - 5 \cdot 0^{2/3} = 0$$
So the curve passes through $(0,0)$.

The x-intercept is
$$x^{5/3} - 5x^{2/3} = 0$$
$$\Rightarrow x^{2/3}(x-5) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 5.$$
So the x-intercept is $x=0$, $x=5$.

c) Symmetry.

Plugging in $-x$ for x gives
$$y(-x) = (-x)^{5/3} - 5(-x)^{2/3}$$
$$= \sqrt[3]{(-x)^5} - 5\sqrt[3]{(-x)^2}$$
$$= \sqrt[3]{-x^5} - 5\sqrt[3]{x^2} = -x^{5/3} - 5x^{2/3}$$

since this isn't equal to either $y(x)$ or $-y(x)$, the function is not even or odd.

D) Asymptotes. Vertical

The function has no vertical asymptotes, the function never goes to $\pm\infty$.

Horizontal: $\lim_{x\to\infty} (x^{5/3} - 5x^{2/3}) = \lim_{x\to\infty} x^{5/3}(1-5x^{-1})$

$$= \lim_{x\to\infty} x^{5/3} \cdot \lim_{x\to\infty}\left(1-\frac{5}{x}\right) = \infty$$

$$\lim_{x\to-\infty} (x^{5/3} - 5x^{2/3}) = \lim_{x\to\infty} x^{5/3}\left(1-\frac{5}{x}\right) = -\infty$$

So there are no horizontal asymptotes.

E) Increasing / Decreasing

We calculate
$$y' = \frac{5x^{2/3} - 10x^{-1/3}}{3}$$

so the critical values $y'=0$ are $\frac{5x^{-1/3}(x-2)}{3} = 0$.

I.e. $\frac{5}{\sqrt[3]{x}} \cdot \frac{x-2}{3} = 0$.

So $x=2$ is a critical value since $y'=0$,

$x=0$ is also a critical value since y' undefined.

Make a quick table:

function	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$x^{-1/3}$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
$y' = f'(x)$	$+$	$-$	$+$
y	$\nearrow$ incr.	$\searrow$ decr.	$\nearrow$ incr.

F) Maxes and mins:

$x=0$ gives a max where $y = 0^{5/3} - 5\,0^{2/3} = 0$

$x=2$ gives a min where $y = 2^{5/3} - 5 \cdot 2^{2/3}$

$= -3 \cdot 2^{2/3}$

G) Concavity / inflection pts.

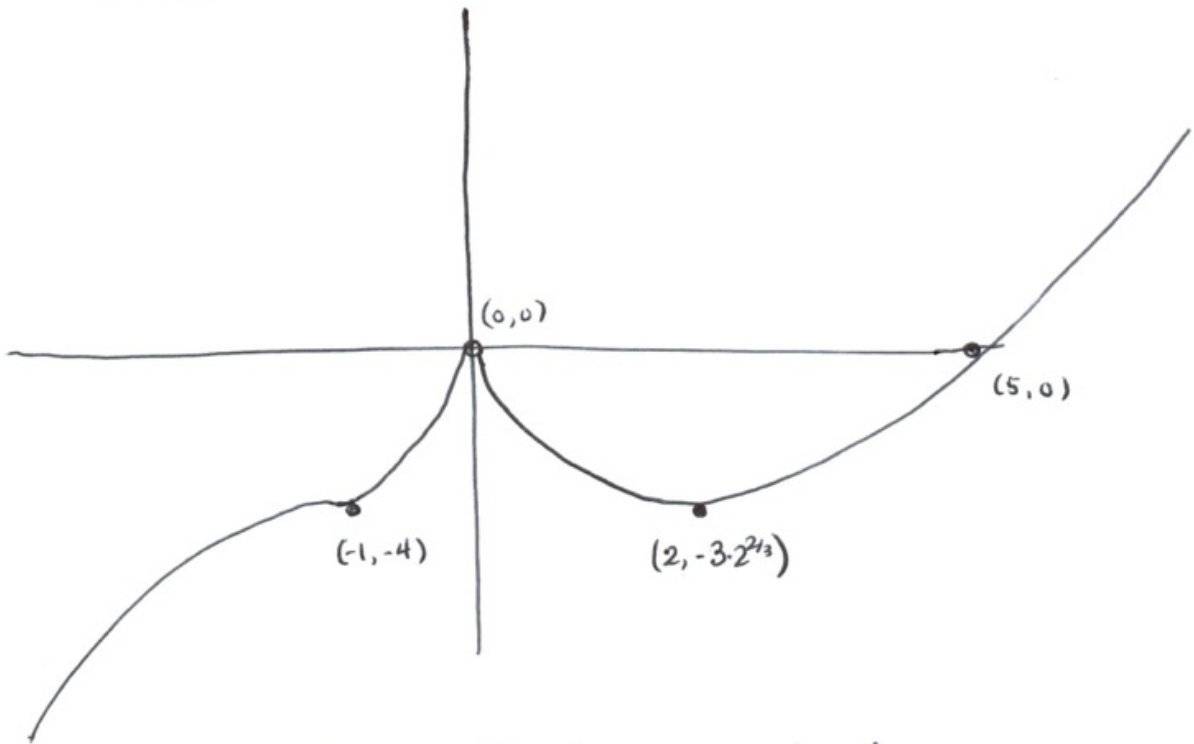
$y''(x) = \dfrac{10x^{-4/3}(x+1)}{9}$, so y'' is undefined at $x=0$ and has a root at $x=-1$.

So $x=0$ and $x=-1$ are potential inflection points.

function	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
$\frac{1}{\sqrt[3]{x^4}} = x^{-4/3}$	$+$	$+$	$+$
$x+1$	$-$	$+$	$+$
$y''(x)$	$-$	$+$	$+$
$y(x)$	$\frown$ conc. down	$\smile$ conc. up	$\smile$ conc. up.

So when $x=-1$ $y=-4$ is an inflection point.

H) Sketch.



Why a point at $x=0$? You can check
$\lim_{x\to 0^-} y'(x) = +\infty$ and $\lim_{x\to 0^+} y'(x) = -\infty$.

Example. Sketch $y = x\sqrt{2-x^2}$.

Solution:

A) We need $2-x^2 \geq 0$ or $x^2 \leq 2$, so x is between $-\sqrt{2}$ and $\sqrt{2}$.

B) y-intercept is $y(0) = 0\cdot\sqrt{2-0^2} = 0$.

x-intercepts are $x\cdot\sqrt{2-x^2} = 0$

$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}.$$

C) We plug in $(-x)$

$$y(-x) = -x\sqrt{2-(-x)^2} = -x\sqrt{2-x^2} = -y(x),$$

so y is odd.

D) There are no vertical asymptotes because y does not go to infinity anywhere.

There are no horizontal asymptotes because x is between $-\sqrt{2}$ and $\sqrt{2}$, so $x \to \pm\infty$ is not possible.

E) Increasing / Decreasing

We calculate $$y' = \frac{2-2x^2}{\sqrt{2-x^2}} = 2\frac{(1-x)(1+x)}{\sqrt{2-x^2}}$$

It is undefined at $x = \pm\sqrt{2}$, the endpoints of the domain. It is zero for $2-2x^2=0$

$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1.$$

function	$(-\sqrt{2},-1)$	$(-1, 1)$	$(1, \sqrt{2})$
$(1-x)$	+	+	−
$(1+x)$	−	+	+
$y'(x)$	−	+	−
$y(x)$	↘	↗	↘
	decr.	incr.	decr.

F) There is a min at $x=-1$ where

$$y = (-1)\sqrt{2-1^2} = (-1)\cdot 1 = -1$$

and a max at $x=1$ where $y = 1\cdot\sqrt{2-1^2} = 1.$

G) The second derivative is

$$y'' = \frac{3x^3 - 6x}{(2-x^2)^{3/2}}$$

This gives inflection points by solving $y''=0$

$\Rightarrow 3x^3 - 6x = 0 \Rightarrow 3x(x^2-2) = 0$

$\Rightarrow x = 0$ or $x = \pm\sqrt{2}$

So $x=0$ is the only potential inflection point since $\pm\sqrt{2}$ are the endpoints of the domain.

Note that

$(2-x^2)^{3/2}$ is always positive, it's the square root of something, and

(x^2-2) is positive when x is in $(-\sqrt{2}, \sqrt{2})$.

So $y'' = \frac{3x(x^2-2)}{(2-x^2)^{3/2}}$ is neg when $x<0$

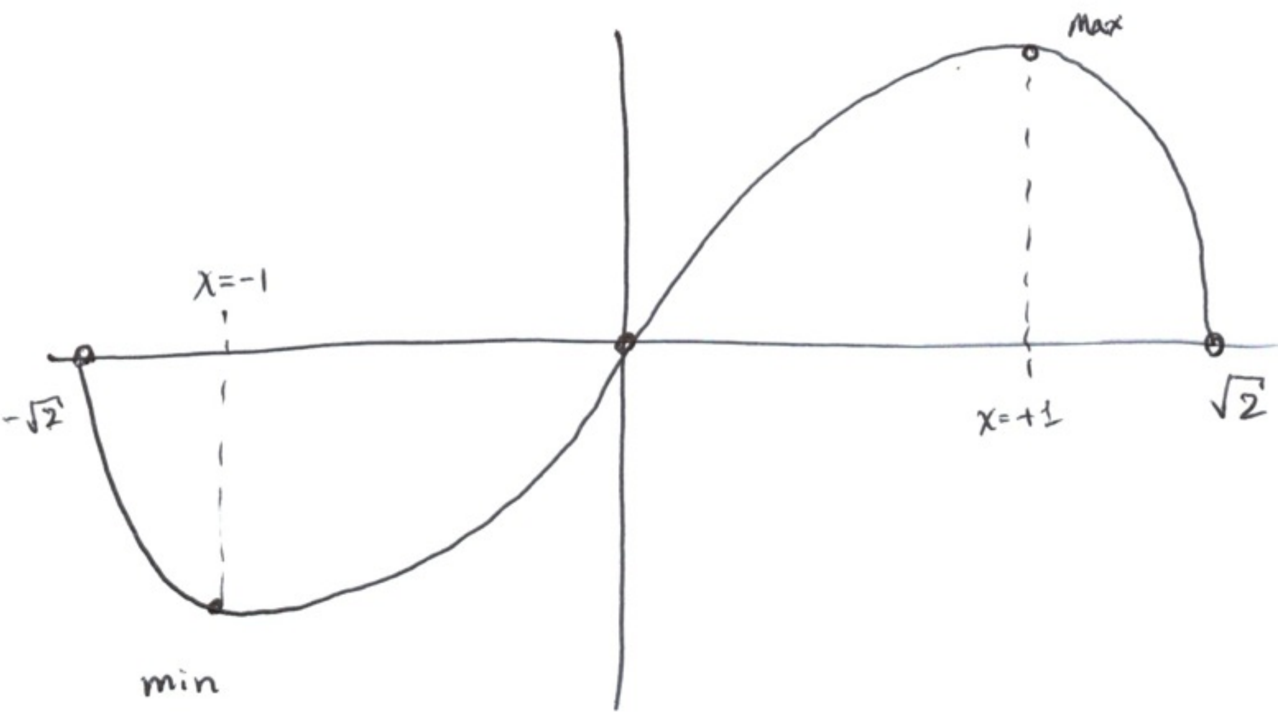
y'' is pos when $x>0$

$\Rightarrow$ $y(x)$ is concave down on $(-\sqrt{2}, 0)$

$y(x)$ is concave up on $(0, \sqrt{2})$.

$(0,0)$ is an inflection point.

H) Sketch.


Max
x=-1
$-\sqrt{2}$
x=+1
$\sqrt{2}$
min

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§4.7 Questions 1-21.

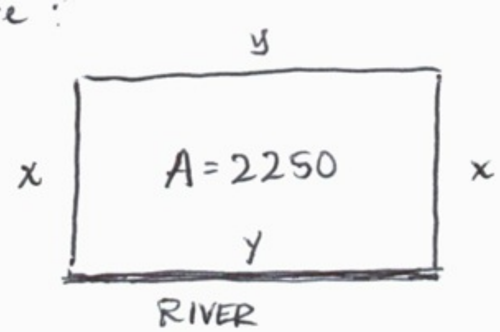
Another type of word problems: Optimization (uses max/min stuff).

As with the previous kind of problems (related rates), there are basically 3 steps:

1. Draw a picture, name all your quantities with variables and list knowns/unknowns.
2. Write an equation relating the quantity Q that you wish to maximize to all the other variables.
3. Differentiate the equation for Q, find its critical values and see which one is the absolute max or absolute min.

Example: A farmer is building a fence to enclose a small pasture of size $2250\,m^2$. The pasture will be a rectangle. Three sides will be made of material that costs \$4/m of fencing, one side along a river must be made of fencing costing \$16/m.
Determine the cheapest way of building the fence.

Solution: Picture:



Need to minimize cost, C.

The cost is $C = 4x + 4x + 4y + 16y$

$= 8x + 20y$

And the lengths x, y must satisfy $xy = A = 2250$,

or $x = \frac{2250}{y}$, $y = \frac{2250}{x}$ (substitute whatever you like).

We plug in $y = \frac{2250}{x}$:

$$C(x) = 8x + 20\left(\frac{2250}{x}\right) = 8x + \frac{45000}{x}$$

Now we set $C'(x) = 0$ to solve for the minimum cost:

$$C'(x) = 8 + (-1)\frac{45000}{x^2} = 8 - \frac{45000}{x^2}.$$

So $$0 = 8 - \frac{45000}{x^2}$$

$$\Rightarrow \frac{45000}{x^2} = 8 \quad \Rightarrow x^2 = \frac{45000}{8} = 5625$$

$$\Rightarrow \underline{\underline{x = 75}}.$$

So $x = 75$ is a critical value. We need to check that it's actually a place of <u>minimum cost</u> not <u>max</u>. We use the second derivative test:

$$C''(x) = 0 - \frac{(-2)\,45000}{x^3} = \frac{45000}{x^3}.$$

So, plugging in $x = 75$ will give a positive number.

$\Rightarrow C(x)$ is concave up

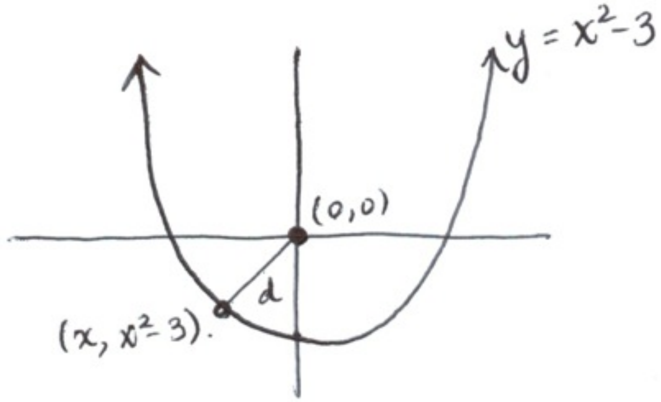
$\Rightarrow x = 75$ is a minimum.

So the cheapest fence has dimensions $x = 75\,m$, $y = \frac{2250}{75} = 30\,m$.

The cost is $C = 8 \cdot 75 + 20 \cdot 30 = \$1200$.

<u>Example</u>: Find the point on the parabola $y = x^2 - 3$ that is closest to the origin.

<u>Solution</u>: The picture is



So the quantity we want to minimize is

$$d = \sqrt{(0-x)^2 + (0-(x^2-3))^2}$$
$$= \sqrt{x^2 + (x^2-3)^2}$$

Here we are using the formula for the distance between two points: (x_0, y_0) and (x_1, y_1):

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

<u>Note</u>: For these distance questions, it's easier to minimize $D = x^2 + (x^2-3)^2$ instead of $d = \sqrt{x^2 + (x^2-3)^2}$

So $D'(x) = 2x + (2x)(x^2-3) \cdot 2$
$= 4x^3 - 10x = x(4x^2 - 10)$.

Thus $D(x)$ has critical values $x=0$ and

$$4x^2-10=0 \Rightarrow x^2 = \frac{10}{4} \Rightarrow x = \pm\sqrt{\frac{10}{4}} = \pm\sqrt{\frac{5}{2}}.$$

We have to determine which is a local max and which is a local min. Try second derivative test:

$D''(x) = 12x^2 - 10$.

Then:

$$D''(0) = -10 \Rightarrow D(x) \text{ concave down}$$
$$\Rightarrow x=0 \text{ is local max.}$$

$$D''\left(\pm\sqrt{\tfrac{5}{2}}\right) = 12\cdot\left(\tfrac{5}{2}\right) - 10 = 30-10=20$$
$$\Rightarrow D(x) \text{ concave up}$$
$$\Rightarrow x = \pm\sqrt{\tfrac{5}{2}} \text{ are both local mins.}$$

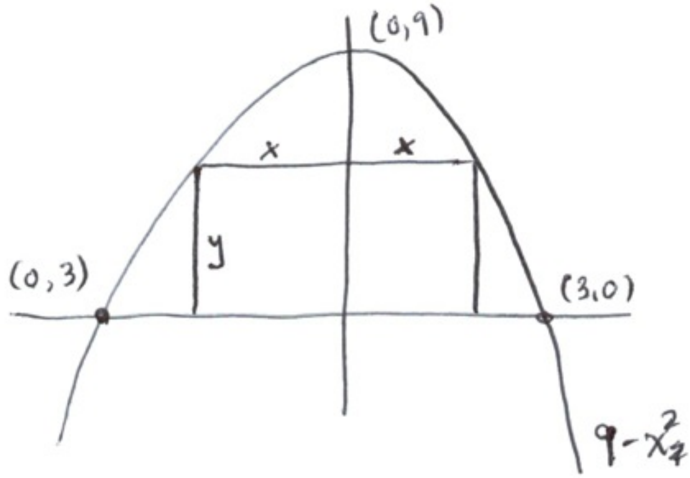
Answer: There are two points on $y=x^2-3$ closest to $(0,0)$, they are $x=\pm\sqrt{\frac{5}{2}}$ and

$$y=\left(\pm\sqrt{\tfrac{5}{2}}\right)^2-3 = \frac{5}{2}-3 = \frac{-1}{2}, \text{ ie.}$$

$$\left(\frac{\sqrt{5}}{2}, -\frac{1}{2}\right) \text{ and } \left(-\frac{\sqrt{5}}{2}, \frac{1}{2}\right).$$

Example. What is the largest area of a rectangle that fits between the parabola $y=9-x^2$ and the x-axis?

Solution: The picture is:



So the area of the rectangle is

$$A = 2xy, \quad y = 9-x^2$$
$$= 2x(9-x^2)$$
$$= 18x - 2x^3.$$

So we try to maximize A. We find

$$A'(x) = 18 - 6x^2$$

so $A'(x) = 0 \Rightarrow 6(3-x^2) = 0$

$\Rightarrow x = \pm\sqrt{3}$, take $+\sqrt{3}$ since it's a distance.

We use the second derivative test to check it's a max:

$$A''(x) = -12x$$
$$= -12 \cdot \sqrt{3} < 0,$$

so $A(x)$ is concave down and $x = \sqrt{3}$ is a max.

So the biggest rectangle is $x = \sqrt{3}$, $y = 9 - (\sqrt{3})^2 = 6$.